# Some Notes on Two–Scale Difference Equations

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**Abstract:** In this paper, we continue our considerations in [2] on two-scale difference equations, mainly with respect to continuous solutions. Moreover, we study refinable step functions and piecewise polynomials. Also, solutions with noncompact support are considered. New algorithms for the approximative computation of continuous solutions are derived.

**Key words:** Refinement equation, piecewise continuous solutions, spectral properties of coefficient matrices, new algorithms
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## 1. INTRODUCTION

A two–scale difference equation is a functional equation of the form

$$\varphi(t/2) = \sum_{\nu=0}^{n} c_\nu \, \varphi(t-\nu), \tag{1.1}$$

where $c_\nu$ are given real or complex constants with $c_0 \, c_n \neq 0$ and $n \geq 1$. A function $\varphi$ satisfying (1.1) for all real $t$ is called *refinable*. Functional equations of type (1.1) arise especially in the construction of wavelets as well as in interpolating subdivision schemes. They are considered in a lot of papers (see e.g. [15, 6, 7, 3, 4, 11, 13, 2]).

If $\varphi \in L^1(\mathbb{R})$ is assumed, then $\varphi$ has necessarily a compact support contained in $[0, n]$ (see [6]). In [2], we have restricted us to this case.
In the present paper, we continue our considerations in [2] on two-scale difference equations, mainly with respect to nonzero continuous solutions $\varphi$ of (1.1).
Functional equations of similar types as in (1.1) occur in physics and can be investigated with related methods. For example, the equation

$$y(qt) = \frac{1}{4q}[y(t+1) + y(t-1) + 2y(t)], \quad (0 < q < 1),\, t \in \mathbb{R},$$

associated with the appearance of spatially chaotic structures in amorphous (glassy) materials, was intensively studied (see e.g. [1, 16, 9, 8]). Systems of such functional equations are useful for applications in probability theory [17] and in the theory of fractal objects (see e.g. [10]).
Usually, the Fourier transform is the main tool for solving this type of equations. By Fourier transform $\hat{\varphi}(u) = \int_{-\infty}^{\infty} \varphi(t)\, e^{-iut} \, \mathrm{d}t$ of (1.1), we obtain

$$\hat{\varphi}(2u) = P(e^{-iu})\, \hat{\varphi}(u) \tag{1.2}$$

with the two–scale symbol (or refinement mask)

$$P(z) := \frac{1}{2} \sum_{\nu=0}^{n} c_\nu\, z^\nu. \tag{1.3}$$

In [6], it is proved for real coefficients $c_\nu$ that:
(i) if $|P(1)| < 1$ or $P(1) = -1$, then (1.1) has no $L^1$-solution;
(ii) if $P(1) = 1$, then it has at most one linearly independent $L^1$-solution;
(iii) if $|P(1)| > 1$, and if a nonzero $L^1$-solution $\varphi$ exists, then $P(1) = 2^m$ for some nonnegative integer $m$. If in the last case, the coefficients $c_\nu$ ($\nu = 0, \ldots, n$) are replaced by $2^{-m} c_\nu$ in (1.1) then the new two-scale difference equation possesses a nonzero integrable solution $g$ such that $\varphi$ is the $m$-th derivative of $g$:

$$\varphi(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} g(x) \quad a.e..$$

Hence, looking for compactly supported solutions, we can essentially restrict us to the case $P(1) = 1$. Then, repeated application of (1.2) yields

$$\hat{\varphi}(u) = \prod_{j=1}^{\infty} P(e^{-iu/2^j}), \tag{1.4}$$

where we assume that $\hat{\varphi}(0) = \int_0^n \varphi(x)\, \mathrm{d}x = 1$ (see e.g. [6]).
Later on, we also shall consider continuous solutions of (1.1), which vanish only for $t \leq 0$, but are polynomials for $t \geq n$. In this case, $P(1) = 2^m$ also for negative integers $m$ is possible.
In the following, if we speak about a solution of (1.1), then we usually mean a continuous one, but not the always existing identically vanishing solution. Some

considerations will be transfered to the case of piecewise contionuous functions and to step functions (see Section 3).

Let us first assume that $\varphi$ is a continuous and compactly supported solution of (1.1). Introducing the $(n+1)$-dimensional vector

$$\psi(t) = (\varphi(t), \varphi(t+1), \ldots, \varphi(t+n))^T \tag{1.5}$$

and the $(n+1)\times(n+1)$-matrix $A := (c_{2i-j})$, $i,j = 0,\ldots,n$, where $c_i = 0$ for $i<0$ and $i>n$, respectively,

$$A := (c_{2i-j})_{i,j=0}^{n} = \begin{pmatrix} c_0 & 0 & 0 & \ldots & 0 \\ c_2 & c_1 & c_0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & c_n & c_{n-1} & c_{n-2} \\ 0 & \ldots & 0 & 0 & c_n \end{pmatrix},$$

(1.1) can be written in vector form

$$\psi(t/2) = A\,\psi(t) \tag{1.6}$$

for $-1 \le t \le 1$. Note that $A$ is a $1\times 2$ block Toeplitz matrix (see e.g. [19]), and that $\psi(t)$ has at most $n$ nonvanishing entries; for $t \le 0$ the first and for $t \ge 0$ the last component vanishes. If the coefficients $c_\nu$ are symmetric, $c_\nu = c_{n-\nu}$ $(\nu = 0,\ldots,n)$, then it easily follows that $\varphi(t) = \varphi(n-t)$. Equation (1.6) implies

$$\psi(2^{-k}t) = A^k\,\psi(t) \tag{1.7}$$

and for $t = 0$

$$\psi(0) = A\,\psi(0). \tag{1.8}$$

According to $\psi(0) \neq 0$ for a nonzero continuous solution (cf. [7], Proposition 2.1), $\psi(0)$ is necessarily a right eigenvector of $A$ corresponding to the eigenvalue 1.
By means of the shifting matrices

$$V := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & 1 \\ 0 & \ldots & 0 & 0 & 0 \end{pmatrix}, \qquad V^T := \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 & 0 \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix},$$

we have for $-1 \le t \le 1$

$$\psi(t+1) = V\,\psi(t), \qquad \psi(t-1) = V^T\,\psi(t). \tag{1.9}$$

In view of $V\,V^T = \operatorname{diag}\,(1,\ldots,1,0)$ and $V^T\,V = \operatorname{diag}\,(0,1,\ldots,1)$, we obtain $\psi(t) = V\,V^T\psi(t)$ for $t \ge 0$ and $\psi(t) = V^T\,V\,\psi(t)$ for $t \le 0$. Replacing $t$ by $t-1$ in (1.6), it follows that

$$\psi\left(\frac{t-1}{2}\right) = A\,\psi(t-1) = A\,V^T\,\psi(t)$$

for $0 \le t \le 1$, and hence, by (1.9),

$$\psi\left(\frac{t+1}{2}\right) = V\,\psi\left(\frac{t-1}{2}\right) = V\,A\,V^T\,\psi(t) \quad (0 \le t \le 1).$$

**Recursive computation of $\varphi(t)$.** For the calculation of a solution $\varphi$ of (1.1) at dyadic rationals, it is convenient to introduce the following $(n \times n)$-submatrices of $A$

$$A_0 := (c_{2i-j-1})_{1\le i,j\le n}, \qquad A_1 := (c_{2i-j})_{1\le i,j\le n},$$

and the $n$-dimensional vector $\tilde{\psi}(t) = (\varphi(t), \varphi(t+1), \ldots, \varphi(t+n-1))^T$ (cf. e.g. [4, 15, 7, 13]). Then starting with a suitable eigenvector $\tilde{\psi}(0)$ of $A_0$ (or with an eigenvector $\tilde{\psi}(1)$ of $A_1$) belonging to the eigenvalue 1, we have to apply consecutively the relations

$$\tilde{\psi}(\frac{t}{2}) = A_0\,\tilde{\psi}(t), \qquad \tilde{\psi}(\frac{t+1}{2}) = A_1\,\tilde{\psi}(t). \tag{1.10}$$

In Section 2, we shall discuss the problem, how to choose $\tilde{\psi}(0)$ among the eigenvectors of $A_0$, if 1 is an eigenvalue of $A_0$ with multiplicity greater than 1.
After having calculated the values $\varphi(2^{-j})$ $(l,\, j \in \mathbb{N})$ by means of (1.10), we can interpolate them by continuous spline functions

$$\varphi_j(t) := \sum_{l=0}^{2^j n} \varphi(2^{-j}l)\,h(2^j t - l) \qquad (j = 0, 1, 2, \ldots), \tag{1.11}$$

where $h(t)$ is the hat function

$$h(t) := \begin{cases} 1 - |t| & |t| \le 1, \\ 0 & |t| > 1. \end{cases}$$

As shown in [6, 15], if there exists a nonzero continuous solution $\varphi$ of (1.1) then $\lim_{j\to\infty} \varphi_j = \varphi$. Note that this dyadic interpolation method is different from the subdivision scheme considered e.g. in [3] and [5], pp. 207. In particular, this method also applies if the integer translates of $\varphi$ are linearly dependent, while the subdivision algorithm does not work in this case (see e.g. [3]).
At this point, we want to mention, that $\tilde{\psi}(t)$ can also be computed at certain nondyadic values in a similar way. From (1.10), it follows that

$$\tilde{\psi}\left(\frac{t+1}{4}\right) = A_0\,A_1\,\tilde{\psi}(t), \quad \tilde{\psi}\left(\frac{t+2}{4}\right) = A_1\,A_0\,\tilde{\psi}(t),$$

and hence

$$\tilde{\psi}\left(\frac{1}{3}\right) = A_0\,A_1\,\tilde{\psi}\left(\frac{1}{3}\right), \quad \tilde{\psi}\left(\frac{2}{3}\right) = A_1\,A_0\,\tilde{\psi}\left(\frac{2}{3}\right).$$

This means that $A_0\,A_1$ and $A_1\,A_0$ must also have the eigenvalue 1, and the vectors $\tilde{\psi}(1/3)$ and $\tilde{\psi}(2/3)$ can be computed as eigenvectors. Generally, we find

$$\tilde{\psi}\left(\frac{i}{2^k-1}\right) = A_{\nu_k}\,A_{\nu_{k-1}}\ldots A_{\nu_1}\,\tilde{\psi}\left(\frac{i}{2^k-1}\right)$$

for $i = 1, \ldots, 2^k - 2$, where the indices $\nu_j \in \{0, 1\}$ are defined by the dyadic representation $i = \sum_{j=1}^{k} \nu_j \, 2^{j-1}$. If we restrict ourselves to simple eigenvalues, the corresponding eigenvectors are determined only up to a normalization factor. In order to find the correct factor, we can use Theorem 2.1 in [2], which gives $\sum_{j=-\infty}^{\infty} \varphi(t+j) = \hat{\varphi}(0) = 1$, which is possible in the case $P(1) = 1$ (see Formula (1.4)). Equivalently, the sum of all components of $\tilde{\psi}(i/(2^k - 1))$ is equal to 1.

Necessary and sufficient conditions for the existence of a nonzero continuous solution of (1.1) are already presented in [15] and [4], involving infinite products of matrices $A_0$ and $A_1$. Further, if (1.1) is assumed to have a nonzero, compactly supported, integrable solution, some necessary conditions for the refinement mask $P(z)$ can be given. In particular, $P(z)$ necessarily contains a polynomial factor $p(z)$, where all zeros of $p(z)$ are roots of $-1$ of order $2^r$ ($r \in \mathbb{N}$) (see [2]). This polynomial factor $p(z)$ can be seen as the refinement mask of a certain step function.
In Section 2, we investigate the structure of the $(n+1) \times (n+1)$-matrix $A$ by means of Jordan's normal form. The goal is to derive simple necessary conditions for the existence of nonzero continuous solutions of (1.1). In Section 3, we present a procedure for the construction of an arbitrary refinement mask of a refinable step function. Polynomial solutions of (1.1) are considered in Section 4. In particular, we show how an integral equation can be solved approximately. In Section 5, we introduce a new algorithm which is based on a factorization of the refinement mask $P(z)$ into a simpler part $\tilde{P}(z)$, being the refinement mask of a known function (e.g. a step function) and a remainder part $Q(z)$,

$$P(z) = \tilde{P}(z) \, Q(z).$$

Assuming that a (piecewise) differentiable solution $\varphi_0$ of (1.1) corresponding to $\tilde{P}(z)$ is given explicitly, we get the desired solution of (1.2) with the original mask $P(z)$ in the form

$$\hat{\varphi}(u) = \hat{\varphi}_0(u) \prod_{k=1}^{\infty} Q(e^{-iu/2^k}).$$

Then $\hat{\varphi}(u)$ can be recursively approximated taking

$$\hat{\varphi}_n(u) = Q(e^{-iu/2^n}) \, \hat{\varphi}_{n-1}(u),$$

and the corresponding original functions $\varphi_n$ converge to a continuous (or even differentiable) solution $\varphi$ of (1.1), if $Q(z)$ behaves well. In particular, this new algorithm also works if the integer translates of $\varphi$ are linearly dependent. Finally, in Section 6, we consider the case of continuous solutions of (1.1) with support $[0, \infty)$ which are polynomials for $t \geq n$.

## 2. JORDAN'S NORMAL FORM

Now, we want to consider the matrix $A$ in detail. Assuming the existence of a nonzero continuous solution $\varphi$ of (1.1), we shall derive a set of simple necessary criterions on $A$.

Let $A = W^{-1}\,J\,W$ be the decomposition of $A$ into Jordan's normal form, where

$$J := \begin{pmatrix} J_0 & & & \\ & J_1 & & \\ & & J_2 & \\ & & & J_3 \end{pmatrix}, \quad W = \begin{pmatrix} W_0^T \\ W_1^T \\ W_2^T \\ W_3^T \end{pmatrix}, \quad W^{-1} = (U_0\,U_1\,U_2\,U_3)\,.$$

Here, $J_i$ contains the Jordan blocks with eigenvalues

$\lambda = 0$ for $i = 0$,

$\lambda = 1$ for $i = 1$,

$|\lambda| < 1$ and $\lambda \neq 0$ for $i = 2$ as well as

$|\lambda| \geq 1$ and $\lambda \neq 1$ for $i = 3$.

If such blocks do not appear, we consider the corresponding matrices as empty. Further, $W_i^T$ are matrices consisting of the rows of $W$, and $U_i$ are the matrices consisting of the columns of $U = W^{-1}$ corresponding to $J_i$. In particular, we have

$$\sum_{i=0}^{3} U_i\,J_i\,W_i^T = A, \qquad \sum_{i=0}^{3} U_i\,W_i^T = I$$

as well as

$$W_i^T\,U_i = I$$

and

$$W_i^T\,U_j = 0$$

for $i \neq j$ $(i, j = 0, 1, 2, 3)$, where $I$ and $0$ denote identity matrices and zero matrices of suitable size, respectively. As shown in [2], Theorem 5.1, if (1.1) possesses a nonzero, continuous $L^1$-solution, then the Jordan block $J_1$ in the Jordan decomposition of $A$ (belonging to the eigenvalue 1) is a nonempty identity matrix, i.e. $J_1 = I$. With the notations above, (1.7) can be written in the form

$$\psi(2^{-k}t) = U\,J^k\,W\,\psi(t). \tag{2.1}$$

Furthermore, for $i = 0,\ 1,\ 2,\ 3$ it follows that

$$W_i^T\,\psi(2^{-k}t) = J_i^k\,W_i^T\,\psi(t) \qquad (k \in \mathbb{N}) \tag{2.2}$$

for $-1 \leq t \leq 1$.

**Theorem 2.1** *If $\psi$ is a nonzero, continuous $L^1$-solution of* (1.6)*, then we have for $-1 \leq t \leq 1$,*

$$W_0^T\,\psi(t) = 0, \qquad W_3^T\,\psi(t) = 0, \qquad W_2^T\,\psi(0) = 0, \tag{2.3}$$

*as well as*

$$W_1^T\,\psi(t) = W_1^T\,\psi(0). \tag{2.4}$$

**Proof:** The convergence of the left-hand side of (2.2) for $k \to \infty$ and the divergence of $J_3^k$ in all elements of the main diagonal implies the relation $W_3^T\,\psi(t) = 0$.
If $A$ has root vectors of height at least $m$ corresponding to the eigenvector 0, then $J_0^m$ is a zero matrix, and (2.2) implies that $W_0^T\psi(t) = 0$ for $-2^{-m} \le t \le 2^{-m}$. For $m = 1$, we obtain our assertion. In the case $m > 1$, we use the following argument: Considering the infinite matrix $\boldsymbol{A} = (c_{2i-j})_{i,j\ge 0}$, it can be shown, that each root vector of $A$ to the eigenvalue 0 can be extendend to a root vector of $\boldsymbol{A}$ corresponding to 0. The assertion $W_0^T\psi(t) = 0$ then follows for all $t \in [-1, 1]$ from [2].
Further, since $J_1 = I$, we get from (2.2) for $j = 1$

$$W_1^T\,\psi(0) = W_1^T\,\psi(t).$$

Finally, for $j = 2$, $k \to \infty$ in Formula (2.2) yields $W_2^T\,\psi(0) = 0$. ∎

**Corollary 2.2** *If $\psi$ is a nonzero, piecewise continuous $L^1$-solution of* (1.6)*, then for $-1 \le t \le 1$ we have* (2.3) *and (instead of* (2.4)*),*

$$W_1^T\,\psi(t) = W_1^T\psi(+0) \ \textit{for}\ t > 0, \qquad W_1^T\,\psi(t) = W_1^T\psi(-0) \ \textit{for}\ t < 0, \tag{2.5}$$

*where both $\psi(+0)$ and $\psi(-0)$ are right eigenvectors of $A$ corresponding to the eigenvalue* 1*.*

This assertion can be shown in the same manner as Theorem 2.1.

**Example 2.3** Consider (1.1) with $c_0 = c_1 = 1$. Then, for arbitrary $a \in \mathbb{R}$,

$$\varphi(t) = \begin{cases} a & t = 0, \\ 1 & t \in (0,1), \\ 1-a & t = 1, \\ 0 & \text{otherwise} \end{cases}$$

is a piecewise continuous solution of (1.1). The eigenvectors of $A = I$ mentioned in Corollary 2.2 are $\psi(+0) = (1,0)^T$ and $\psi(-0) = (0,1)^T$.

Applying Theorem 2.1, we have (for continuous solutions) from (1.6)

$$\psi(\frac{t}{2}) = A\,\psi(t) = \sum_{i=0}^{3} U_i\,J_i\,W_i^T\,\psi(t) = U_1\,W_1^T\,\psi(0) + U_2\,J_2\,W_2^T\,\psi(t), \tag{2.6}$$

and in particular for $t = 0$,

$$\psi(0) = U_1\,W_1^T\,\psi(0). \tag{2.7}$$

Hence,

$$\psi(\frac{t}{2}) = \psi(0) + U_2 J_2 W_2^T \psi(t). \tag{2.8}$$

A lot of papers dealing with two–scale difference equations (cf. e.g. [7, 12]), are especially interested in nonzero continuous solutions of (1.1) with linearly independent integer translates $\varphi(t+l)$ ($l \in \mathbb{Z}$). Here, we say that $\varphi(t+l)$ are *linearly independent*, if, for all finite linear combinations, $\sum_l a_l \varphi(t+l) = 0$ implies that $a_l = 0$ for all $l$. The linear independence (or at least $L^2$–stability (see e.g. [11])) of $\varphi(t+l)$ ($l \in \mathbb{Z}$) is crucial for applications of $\varphi$ as a scaling function in wavelet theory as well as for the convergence of the corresponding subdivision scheme (cf. [11]). In this case we have:

**Lemma 2.4** *Let* (1.1) *possess a nonzero continuous $L^1$-solution $\varphi$ with linearly independent integer translates. Then both, $J_3$ and $J_0$, are empty.*

**Proof:** If $\varphi$ satisfies the assumptions of the lemma, then the vectors $\psi(t)$ ($t \in [-1,1]$) span the whole $\mathbb{R}^{n+1}$, i.e., we have

$$\operatorname{span}\{\psi(t) : t \in [-1,1]\} = \mathbb{R}^{n+1}.$$

If $W_0^T$ and $W_3^T$ would not be empty, this would be a contradiction to (2.3). ∎

We recall that the condition of linear independence for $\varphi$ is equivalent to the condition that the Casorati determinant

$$\det \begin{pmatrix} \varphi(t_1) & \varphi(t_2) & \dots & \varphi(t_n) \\ \varphi(t_1+1) & \varphi(t_2+1) & \dots & \varphi(t_n+1) \\ \vdots & \vdots & & \vdots \\ \varphi(t_1+n-1) & \varphi(t_2+n-1) & \dots & \varphi(t_n+n-1) \end{pmatrix}$$

does not vanish for any $t_k \in [0,1]$ ($k = 1, \dots, n$) with $t_k \neq t_j$ for $k \neq j$ (cf. [18]). This is satisfied if and only if e.g. the two conditions

(a) $P(z)$ has no symmetric zeros in $\mathbb{C} \setminus \{0\}$ (i.e., if $z_0 \neq 0$ is a zero, then $-z_0$ is no zero);

(b) For any odd integer $m > 1$ and any primitive $m$th root $\omega$ of unity, there exists an integer $d \geq 0$ such that $P(-\omega^{2^d}) \neq 0$;

are satisfied (cf. [11, 12]).

**Computation of $\psi(0)$.** For application of the vector cascade algorithm (see in the Introduction), we first need to compute $\psi(0)$. If the eigenvalue 1 of A is simple (as e.g. assumed in [7]), then $u := U_1$ is a vector (of length $n+1$) and $\psi(0)$ is necessarily determined by $u$ up to normalization. But, how to choose the initial vector $\psi(0)$, if $\dim J_1 > 1$? Introducing the vector

$$x := W_1^T \, \psi(0), \tag{2.9}$$

we find from (2.7) that $\psi(0) = U_1 \, x$. Using (2.4), it follows that $x = W_1^T \, \psi(t)$, and by $\psi(1) = V \, \psi(0)$ and $\psi(-1) = V^T \, \psi(0)$, we obtain

$$x = W_1^T \, V^T \, U_1 \, x = W_1^T \, V \, U_1 \, x. \tag{2.10}$$

A similar consideration can be done with respect to (2.3) yielding

$$W_i^T\, V\, U_1 x = W_i^T\, V^T\, U_1\, x = 0 \tag{2.11}$$

for $i = 0$ and $i = 3$. Hence we have

**Proposition 2.5** *If* (2.10)–(2.11) *has only the trivial solution $x$, then* (1.1) *possesses only the trivial solution $\varphi(t) \equiv 0$ as continuous solution. If* (1.1) *possesses a nonzero continuous solution then $x$ in* (2.9) *satisfies* (2.10) *and* (2.11).

The importance of the vector $x$ lies in the fact, that the initial vector $\psi(0) = U_1\, x$ needed for the numerical computation of $\psi(t)$ is known, if we know $x$. Computational observations lead us to the following

**Conjecture.** *If a nonzero, continuous solution of* (1.1) *exists, then $x$ is already uniquely determined by the first equation in* (2.10).

**Examples 2.6 (i)** Let (1.1) be given with the coefficients $c_0 = c_6 = 1/2,\ c_3 = 1,\ c_1 = c_2 = c_3 = c_4 = 0$. Then the corresponding coefficient matrix $A$ possesses the double eigenvalue 1 with

$$W_1^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad U_1 = \frac{1}{6}\begin{pmatrix} 0 & 1 & 2 & 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 6 & -2 & -1 & 0 \end{pmatrix}^T.$$

Using (2.10), we obtain that

$$x = \begin{pmatrix} 1 & 0 \\ 1/2 & -1/2 \end{pmatrix} x$$

and hence $x = (3, 1)^T$ up to normalization. By Proposition 2.5, only this choice of $x$ can lead to a continuous solution of (1.1). The starting vector $\psi(0)$ is now given by $\psi(0) = (0,\ 1/3,\ 2/3,\ 1,\ 2/3,\ 1/3,\ 0)$ up to normalization. The corresponding solution is

$$\varphi(t) = \begin{cases} t/3 & 0 \le t \le 3, \\ (6-t)/3 & 3 < t \le 6, \\ 0 & \text{otherwise.} \end{cases}$$

**(ii)** Consider the two–scale difference equation corresponding to the refinement mask $P(z) = \frac{1}{2}(x^3+1)^2(x^5+1)^2 = x^{16} + 2x^{13} + 2x^{11} + x^{10} + 4x^8 + x^6 + 2x^5 + 2x^3 + 1$. Then the matrix $A$ possesses the eigenvalue 1 with multiplicity 3, and we find

$$W_1^T =$$

$$\frac{1}{180}\begin{pmatrix} 20 & 14 & 17 & 8 & -1 & 8 & -4 & -7 & 8 & -7 & -4 & 8 & -1 & 8 & 17 & 14 & 20 \\ 30 & 30 & -60 & 30 & 30 & -60 & 30 & 30 & -60 & 30 & 30 & -60 & 30 & 30 & -60 & 30 & 30 \\ 0 & 0 & 45 & -90 & 45 & 0 & 0 & 45 & -90 & 45 & 0 & 0 & 45 & -90 & 45 & 0 & 0 \end{pmatrix},$$

$$U_1^T = \begin{pmatrix} 0 & 1 & 2 & 3 & 2 & 1 & -2 & -4 & -6 & -4 & -2 & 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $W_1^T$ contains a 3-periodic vector in the second and a 5-periodic vector in the third line, whereas the first line is a linear combination of the other two and the eigenvector with the quadratic entries $3(j-8)^2-17$ for $j=0,\ldots,16$. Equation (2.10) leads to

$$x = \frac{1}{60}\begin{pmatrix} 60 & 5 & 3 \\ 0 & -30 & 0 \\ 0 & 0 & -15 \end{pmatrix} x,$$

and we find $x=(1,0,0)^T$. The starting vector $\psi(0)$ is then given by $\psi(0) = (0,1,2,3,2,1,-2,-4,-6,-4,-2,1,2,3,2,1,0)^T$ up to normalization. As a solution, we obtain

$$\varphi(t) = \begin{cases} 0 & t \le 0, \\ t & t \in [0,3], \\ 6-t & t \in [3,5], \\ 16-3t & t \in [5,6], \\ 10-2t & t \in [6,8], \\ \varphi(16-t) & t \ge 8. \end{cases}$$

## 3. REFINABLE STEP FUNCTIONS

In this section, we want to deal with the case of refinable step functions. The results obtained here can be simply transmitted to piecewise polynomial solutions by integration and to integrable solutions by convolution. Observe that the case of refinable spline functions was also studied in [14]. However, this section has another intention. We show that for each compactly supported integrable solution $\varphi$ of (1.1) the corresponding refinement mask $P(z)$ contains a polynomial factor, which itself can be seen as the refinement mask of a step function. In the second part of this section we give a general method for the construction of refinement masks corresponding to step functions.

First let us recall the following result (see [2], Theorems 3.3 and 3.4):

**Theorem 3.1 ([2])** *Assume that* (1.1) *with* $P(1)=1$, *and* $c_0 \neq 1$, $c_n \neq 1$ *possesses a nonzero, Lebesgue–integrable, compactly supported solution* $\varphi$. *Then* $P(z)$ *has a polynomial factor* $p(z)$ *of the form*

$$p(z) = \frac{q(z^2)}{2\,q(z)}. \tag{3.1}$$

*Here* $q(z)$ *is a polynomial of the same degree as* $p(z)$ *and possesses a set* $S$ *of zeros with the following property:* $S$ *contains roots of unity with powers of* 2 *as root exponent, and it is closed regarding to the operation* $z \to z^2$ *(i.e., for* $z \in S$ *it follows that* $z^2 \in S$*). Moreover, denoting the zero set of* $p(z)$ *by* $R$ *and introducing the set* $\tilde{S}$ *of all square roots of elements of* $S$,

$$\tilde{S} := \{z : z^2 \in S\},$$

*we have the relations:*

$$\begin{aligned} R &= \tilde{S} \setminus S, \\ S &= \{z^{2^j} : j \in \mathbb{N},\, z \in R\}. \end{aligned}$$

We want to show that the factor $p(z)$ in (3.1) can be considered as a refinement mask of a special compactly supported step function. Observe that for nonzero step functions, $c_0 = c_n = 1$ is satisfied, since for $t \to +0$, (1.1) yields $\varphi(+0) = c_0\varphi(+0)$, for $t \to n-0$ we find $\varphi(n-0) = c_n\varphi(n-0)$, and $\varphi(+0) \neq 0$, $\varphi(n-0) \neq 0$ must be satisfied. The values of $\varphi(t)$ at integers $t$ can be different from the one-sided limits $\varphi(t+0)$ and $\varphi(t-0)$ (see Example 2.3). Let

$$\varphi(t) = \sum_{\nu=0}^{k} b_\nu \, \chi(t-\nu) \qquad (t \in \mathbb{R}) \tag{3.2}$$

with $b_0 b_k \neq 0$ be a step function, where $\chi(t)$ is the characteristic function of the interval $[0,1)$, i.e., $\chi(t) = \begin{cases} 1 & t \in [0,1), \\ 0 & t \notin [0,1). \end{cases}$ Then we have:

**Theorem 3.2** *If $\varphi$ of the form (3.2) is refinable, then the corresponding refinement mask $P(z)$ is of the form $2P(z) = q(z^2)/q(z)$, where $q(z) := (z-1)(\sum_{\nu=0}^{k} b_\nu z^\nu)$ with the coefficients $b_\nu$ as in (3.2). Moreover, the zero set $\{z \in \mathbb{C} : q(z) = 0\}$ is closed regarding to the operation $z \to z^2$.*

**Proof:** Observe that $\hat{\chi}(u) = (1 - e^{-iu})/iu$. Hence,

$$\hat{\chi}(2u) = \left(\frac{1 + e^{-iu}}{2}\right) \hat{\chi}(u),$$

i.e., $\chi$ is refinable with the mask $(1+z)/2$. Let

$$\tilde{q}(z) = \sum_{\nu=0}^{k} b_\nu \, z^\nu,$$

such that $q(z) = (z-1)\,\tilde{q}(z)$. By Fourier transform of (3.2), we obtain

$$\hat{\varphi}(u) = \tilde{q}(e^{-iu}) \frac{(1 - e^{-iu})}{iu}.$$

Since $\varphi$ is refinable, there is a corresponding refinement mask $P(z)$ with $\hat{\varphi}(2u) = P(e^{-iu})\hat{\varphi}(u)$. Hence

$$\tilde{q}(e^{-2iu}) \frac{(1 - e^{-2iu})}{2iu} = P(e^{-iu})\, \tilde{q}(e^{-iu}) \frac{(1 - e^{-iu})}{iu}.$$

Putting $z := e^{-iu}$ and remembering that $q(z) := \tilde{q}(z)\,(z-1)$, we find

$$P(z) = \frac{1}{2} \, \frac{q(z^2)}{q(z)}. \tag{3.3}$$

Now, let $q(z) = \prod_{\nu=0}^{k}(z - \alpha_\nu)$, where $\alpha_0 = 1$ since $q(1) = 0$. Then,

$$2\,P(z) = \prod_{\nu=0}^{k} \frac{(z - \sqrt{\alpha_\nu})(z + \sqrt{\alpha_\nu})}{(z - \alpha_\nu)}.$$

Hence, for all $\nu = 0, \ldots, k$, there are $\beta_\nu \in \{0, 1\}$, $\gamma_\nu \in \{0, \ldots, k\}$ such that $\alpha_\nu = (-1)^{\beta_\nu}\sqrt{\alpha_{\gamma_\nu}}$ that means: $\alpha_\nu^2 = \alpha_{\gamma_\nu}$. It follows that the set $\{\alpha_0, \ldots, \alpha_k\}$ must be closed regarding to the operation $z \to z^2$, i.e.,

$$\{\alpha_0, \ldots, \alpha_k\} = \{\alpha_\nu^{2^j} : j \in \mathbb{N}_0,\ \nu = 0, \ldots, k\}. \qquad \blacksquare$$

**Remarks: 1.** For piecewise polynomial refinable functions $\varphi$, similar relations for the corresponding refinement mask $P(z)$ as in Theorem 3.2 can be observed (see [14]).
**2.** The zero set $\{\alpha_\nu : \nu = 0, \ldots, k\}$ of $q(z)$ in Theorem 3.2 has the following property: For each $\nu$, there exist integers $m$, $k_\nu \geq 0$ and $l \geq 1$ such that $\alpha_{k_\nu} = \alpha_\nu^{2^m} = \alpha_\nu^{2^{m+l}}$, i.e., $\alpha_{k_\nu} = \alpha_{k_\nu}^{2^l}$. So, $\alpha_{k_\nu}$ is a $(2^l - 1)$st root of unity and $\alpha_\nu$ a $(2^m(2^l - 1))$th root of unity. In view of Euler's Theorem we have $2^{\varphi(\mu)} \equiv 1 \bmod \mu$ for every odd $\mu$, where here $\varphi(\mu)$ denotes the well–known Euler function (cf. e.g. [21]). Since every integer can be represented as $2^m\mu$ with odd $\mu$, it follows that roots of unity of arbitrary order $2^m\mu$ can appear with $l$ at most equal to $\varphi(\mu)$.
**3.** Since only the quotient $q(z^2)/q(z)$ is needed in Theorem 3.2, we can assume without loss of generality that $\tilde{q}(1) = 1$ as far as $\tilde{q}(1) \neq 0$. If indeed $\tilde{q}(1) = 0$, then we can find a factorization $\tilde{q}(z) = (z - 1)^l \tilde{r}(z)$ with some $l \in \mathbb{N}$ and with $\tilde{r}(1) \neq 0$ (and without loss of generality $\tilde{r}(1) = 1$). In this case, it follows for the refinement mask $P(z) = q(z^2)/(2q(z))$ that $P(1) = 2^l$.

How to construct a polynomial $q(z)$ such that $2P(z) = q(z^2)/q(z)$ is a polynomial? The zeros of such a $q(z)$ can be obtained by the following procedure:
We choose an arbitrary finite set $I$ of positive rationals $p_\nu/q_\nu$ containing 1 and consider

$$C_I := \{e^{2\pi i p_\nu/q_\nu} : p_\nu/q_\nu \in I\}.$$

Then we form the closure $\overline{C_I}$ of $C_I$ such that

$$\overline{C_I} := \{z^{2^j} : z \in C_I,\ j \in \mathbb{N}_0\} = C_{\overline{I}},$$

where $\overline{I}$ contains all rationals $p_\nu/q_\nu$ with $e^{2\pi i p_\nu/q_\nu} \in \overline{C_I}$. Note that $\overline{C_I}$ is finite as well since $\{e^{2^{j+1}\pi i p_\nu/q_\nu}\}_{j\in\mathbb{N}_0}$ is a sequence with at most $q_\nu$ different entries. The polynomial with the zero set $\overline{C_I}$ is the desired $q(z)$, and by (3.3), we can compute the refinement mask $P(z)$.

**Example 3.3** Choose $I := \{1, 1/4,\ 1/14\}$ such that

$$C_I = \{e^{2\pi i},\ e^{2\pi i/4},\ e^{2\pi i/14}\}.$$

Forming the closure $C_I$, we find

$$\overline{I} = \left\{ \frac{1}{4}, \frac{1}{2}, 1, \frac{1}{14}, \frac{1}{7}, \frac{2}{7}, \frac{4}{7} \right\},$$

so that the corresponding polynomial $q(z)$ has the degree 7. Hence, the zeros of the corresponding refinement mask $P(z) = q(z^2)/(2q(z))$ are given by the set of rationals

$$K = \left\{ \frac{1}{8}, \frac{5}{8}, \frac{3}{4}, \frac{1}{28}, \frac{15}{28}, \frac{9}{14}, \frac{11}{14} \right\}.$$

These rationals are the numbers below the rationals of $\overline{I}$ in the directed graphs in Figure 1, which present the squares in direction of the arrows.

## 4. POLYNOMIAL SOLUTIONS

We assume now that (1.1) has a compactly supported $m$-times continuously differentiable solution, i.e., $\varphi \in C^m$ for an $m \geq 1$. Then $A$ has the eigenvalues $2^{-\nu}$ ($\nu = 0, \ldots, m$) (cf. [7, 15]). Let $w_\nu^T$, $u_\nu$ be corresponding left and right eigenvectors of $A$, respectively (in the case of simple eigenvalues). By (1.6), we find $\psi^{(\nu)}(t/2) = 2^\nu A\, \psi^{(\nu)}(t)$ ($\nu = 0, \ldots, m$). Applying the second equality of (2.3) to $\psi^{(\mu)}(t)$ (instead of $\psi(t)$) it follows for $0 \leq \nu < \mu \leq m$ that

$$w_\nu^T\, \psi^{(\mu)}(t) = 0 \qquad (t \in [-1, 1]). \tag{4.1}$$

Further, (2.4) yields

$$w_\mu^T\, \psi^{(\mu)}(t) = w_\mu^T\, \psi^{(\mu)}(0) \quad (t \in [-1, 1]). \tag{4.2}$$

Since, as before (see Theorem 2.1), $w^T\, \psi^{(\mu)}(0) = 0$ for all row vectors of $W$ with $w \neq w_\mu$, we obtain $\psi^{(\mu)}(0) = u_\mu\, w_\mu^T\, \psi^{(\mu)}(0)$, analogously as in (2.7). Moreover, by integration of (4.2), it follows that

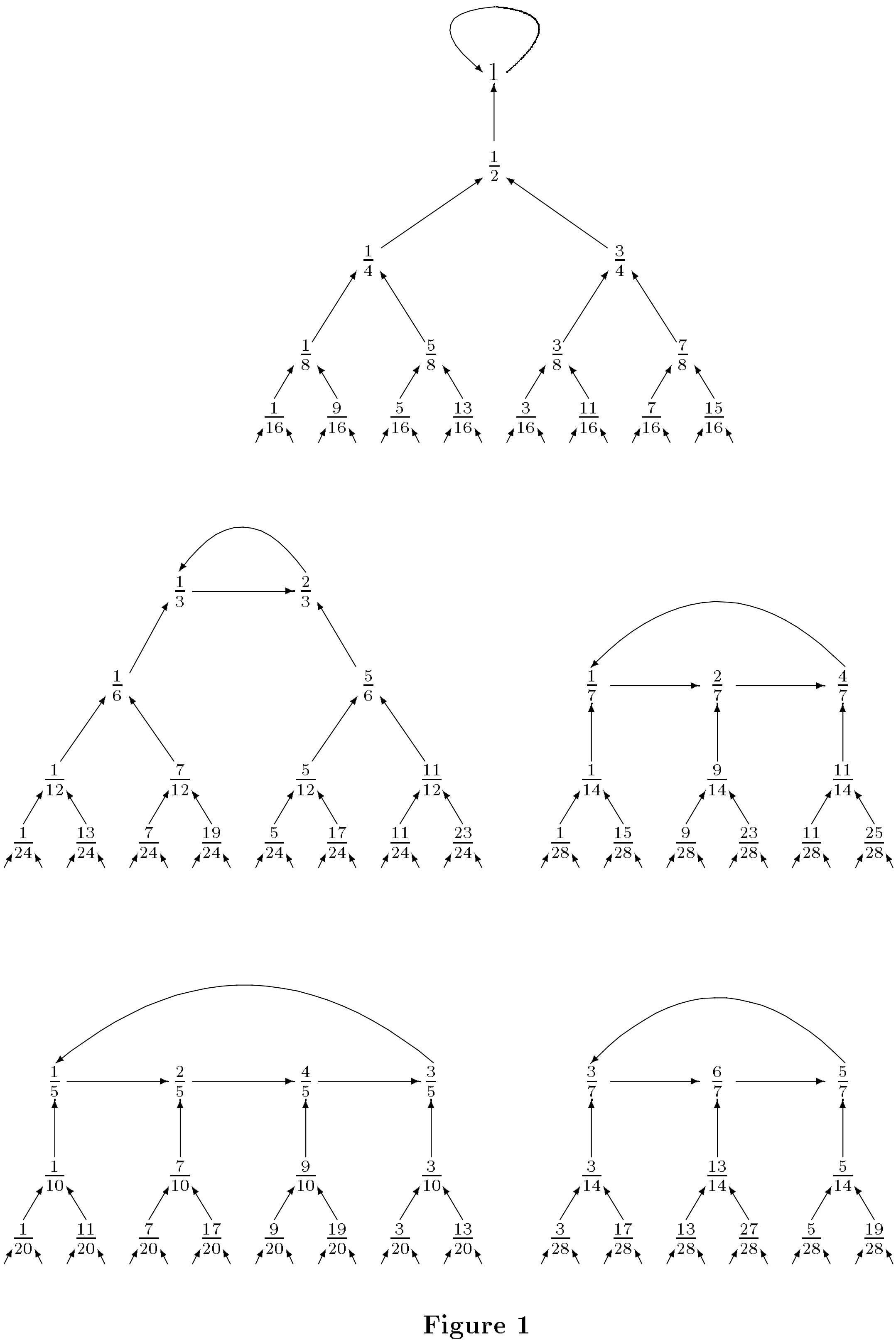
$$w_\mu^T\, \psi^{(\mu-\nu)}(t) = \frac{t^\nu}{\nu!}\, w_\mu^T\, \psi^{(\mu)}(0)$$

for $\nu = 0, \ldots, \mu \leq m$. In particular,

$$w_\mu^T\, \psi(t) = \frac{t^\mu}{\mu!}\, w_\mu^T\, \psi^{(\mu)}(0). \tag{4.3}$$

If $A$ also possesses the eigenvalue $2^{-m-1}$, and if $\varphi^{(m+1)}(t)$ is piecewise continuous, then we also have

$$w_{m+1}^T \psi(t) = \frac{t^{m+1}}{(m+1)!}\, w_{m+1}^T\, \psi^{(m+1)}(+0) \tag{4.4}$$


1
1/2
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1/16 9/16 5/16 13/16 3/16 11/16 7/16 15/16
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1/28 15/28 9/28 23/28 11/28 25/28
1/5 2/5 4/5 3/5
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3/7 6/7 5/7
3/14 13/14 5/14
3/28 17/28 13/28 27/28 5/28 19/28

**Figure 1**

for $t > 0$. The equations (4.3) and (4.4) can be used for a further simplification of (2.8). If we have $n$ linearly independent relations of this kind, then $\varphi(t)$ is uniquely determined by them, and $\varphi(t)$ is a polynomial spline.

**Remarks: 1.** In the case that $2^{-\mu}$ is a multiple eigenvalue, it is possible that $w_\mu^T\,\psi^{(\mu)}(0) = 0$ for a certain $\mu$. This is a new explanation for the fact that also eigenvectors $w$ of $A$ to eigenvalues $\lambda \neq 0$ with $|\lambda| < 1$ can satisfy $w^T\psi(t) = 0$ (see the first remarks in Section 2 in [2]).

**2.** According to Theorem 2.1, equations of the form (4.1) are also valid for left eigenvectors $w^T$ corresponding to all eigenvalues $\lambda \neq 2^{-\mu}$ with $|\lambda| \geq 2^{-\mu}$ and $\mu \leq m$. By integration, we obtain $w^T\,\psi(t) = 0$, since, by (2.3), $w^T$ is orthogonal to all eigenvectors $\psi^{(\kappa)}(0)$ with $0 \leq \kappa \leq \mu$.

**Example 4.1** For the refinement mask $P(z) = \frac{1}{8}(1+z)^2(z^2+1) = \frac{1}{8}(z^4 + 2z^3 + 2z^2 + 2z + 1)$, we obtain

$$A = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = U \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} W^T$$

with

$$W^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1/2 & 0 & -1/2 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}, \qquad U = \frac{1}{4}\begin{pmatrix} 0 & 0 & 4 & 0 & 0 \\ 1 & 4 & -4 & 4 & -1 \\ 2 & 0 & -4 & -4 & 2 \\ 1 & -4 & 4 & -4 & -1 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}.$$

Let $W = (\omega_1,\, \omega_2,\, \omega_3,\, \omega_4,\, \omega_5)$ and $U = (v_1,\, v_2,\, v_3,\, v_4,\, v_5)$, where $\omega_\nu^T$ is the left eigenvector, and $v_\nu$ the right eigenvector of $A$ corresponding to the eigenvalue $\lambda_\nu$ ($\nu = 1, \ldots, 5$) with $\lambda_1 = 1$, $\lambda_2 = 1/2$, $\lambda_3 = \lambda_4 = 1/4$, $\lambda_5 = 0$. Assume that (1.1) has a nonzero, compactly supported, continuous solution with a piecewise continuous second derivative $\varphi''(t)$. Let $\psi(t)$ be the vector (1.5), so that $\psi(\frac{t}{2}) = A\psi(t)$ for $t \in [-1, 1]$. Consider $\psi''(t)$ satisfying $\psi''(\frac{t}{2}) = 4\,A\,\psi''(t)$ for $t \in [0,1]$ almost everywhere. Then, $\psi''(+0)$ is a right eigenvector of $A$ corresponding to the eigenvalue $1/4$. Hence, $\psi''(+0) = C_0\,v_3 = C_0\,(1,\,-1,\,-1,\,1,0)^T$ with some constant $C_0$, since $\varphi''(4+0) = 0$, so that the other right eigenvector $v_4$ of $A$ to the eigenvalue $1/4$ falls out. Formulas (4.1) and (2.5) imply that $\omega_1^T\,\psi''(t) = \omega_2^T\,\psi''(t) = 0$ and $\omega_3^T\,\psi''(t) = \omega_3^T\,\psi''(+0) = C_0$ as well as $\omega_4^T\,\psi''(t) = \omega_4^T\psi''(+0) = 0$. Finally, the first equality in (2.3) yields $\omega_5^T\,\psi''(t) = 0$. Hence, we have

$$W^T\,\psi''(t) = C_0\,(0,\,0,\,1,\,0,\,0)^T$$

leading to $\psi''(t) = C_0\,U\,(0,\,0,\,1,\,0,\,0)^T = C_0\,(1,\,-1,\,-1,\,1,\,0)^T$. Further, $\psi'(0)$ and $\psi(0)$ are right eigenvectors of $A$ corresponding to $\lambda_2 = 1/2$, and $\lambda_1 = 1$, respectively, i.e., $\psi'(0) = C_1 v_2 = C_1 (0,\, 1,\, 0,\, -1, 0)^T$ and $\psi(0) = 4\, C_2\, v_1 = C_2\,(0,\, 1,\, 2,\, 1,\, 0)^T$ with some constants $C_1$, $C_2$. Thus, integration of $\psi''(t)$ leads to $\psi'(t) = (C_0 t,\, C_1 - C_0 t,\, -C_0 t,\, -C_1 + C_0 t,\, 0)^T$ and

$$\psi(t) = (C_0 t^2/2, C_2 + C_1 t - C_0 t^2/2,\, 2C_2 - C_0 t^2/2,\, C_2 - C_1 t + C_0 t^2/2,\, 0)^T.$$

Finally, the continuity of $\varphi$ implies that $C_0 = C_1 = 2C_2$, such that $\psi(t) = C_2(t^2,\, 1 + 2t - t^2,\, 2 - t^2,\, 1 - 2t + t^2,\, 0)^T$. It can easily be checked that $\psi(t)$ really satisfies (1.6). Of course, there are simpler methods to calculate $\psi$.

Polynomial solutions on the whole line $\mathbb{R}$ are considered in Section 6.

**Interval of constancy.** There exist refinable functions, which are polynomials in a certain interval, but not in other intervals. We shall show this for the case of constant polynomials:
Let $\varphi$ be a nonzero, Lebesgue–integrable and compactly supported refinable function with the refinement mask $P(z) = Q(z)\,(z+1)$, $Q(1) = 1/2$ which is normalized by $\hat{\varphi}(0) = 1$. Further, let $\phi$ be the refinable function with the mask $\tilde{P}(z) = Q(z)\,(z^m + 1)$, $m \geq 2$, so that

$$\phi(t) = \sum_{\nu=0}^{m-1} \varphi(t-\nu)$$

according to Theorem 3.9 from [2] with $l = 1$.
Obviously, $Q(z)$ is a polynomial of degree $n-1$, $\operatorname{supp}\varphi = [0,\, n]$, and we have $\sum_{\nu=-\infty}^{\infty} \varphi(t-\nu) = \sum_{\nu=0}^{n-1} \varphi(t-\nu) = \hat{\varphi}(0) = 1$ for $n-1 \leq t \leq n$ (see Theorem 2.1 in [2]). Hence, if $n \leq m$, then it follows that $\sum_{\nu=0}^{m-1} \varphi(t-\nu) = 1$ for $n-1 \leq t \leq m$, i.e., $\phi(t) = 1$ for $n-1 \leq t \leq m$.
For example, let $Q(z) := (2z+1)/6$ and $m = 2$, and consider the compactly supported solutions $\varphi(t)$ and $\phi(t)$ corresponding to the refinement masks $P(z) = \frac{1}{6}(2z{+}1)\,(z{+}1)$ and $\tilde{P}(z) = \frac{1}{6}(2z{+}1)(z^2{+}1)$. Then, supp $\varphi = [0,2]$, supp $\phi(t) = [0,3]$ and, in particular, $\phi(t) = 1$ for $1 \leq t \leq 2$ (see Figure 2).

**An integral equation.** Two–scale difference equations also appear as approximations of certain integral equations. Let us consider the following problem

$$f(\frac{s}{2}) = 2\int_{s-1}^{s} f(t)\ \mathrm{d}t, \quad \int_0^1 f(t)\ \mathrm{d}t = 1, \quad \operatorname{supp} f \subset [0,1] \tag{4.5}$$

which was studied in [20] in differentiated form. A solution of (4.5) obviously satisfies $f \in C^\infty(\mathbb{R})$. A simple consequence of (4.5) is $f(\frac{1}{2}) = 2$. Applying the trapezoidal rule to the first integral, we find

$$f(\frac{s}{2}) = \frac{1}{n}\left( f(s) + f(s-1) + 2\sum_{\nu=1}^{n-1} f(s - \frac{\nu}{n})\right).$$

Putting $s = t/n$ and $\varphi(t) = f(t/n)$, we obtain

$$\varphi(\frac{t}{2}) = \frac{1}{n}\left( \varphi(t) + \varphi(t-n) + 2\sum_{\nu=1}^{n-1} \varphi(t-\nu)\right). \tag{4.6}$$

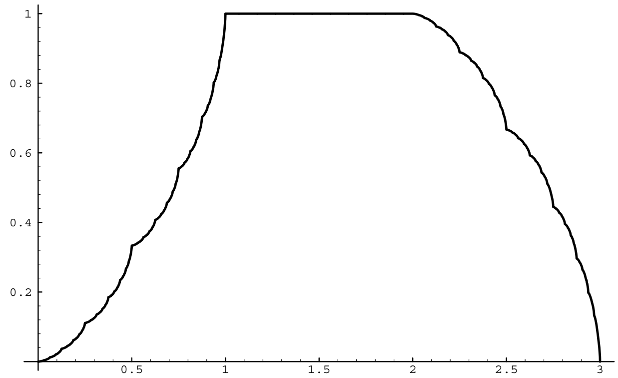


**Figure 2:** Solution $\phi(t)$ of (1.1) with $c_0 = c_2 = 1/3$, $c_1 = c_3 = 2/3$

This is a two-scale difference equation of type (1.1) with the coefficients $c_0 = c_n = 1/n$ and $c_\nu = 2/n$ for $\nu = 1, \ldots, n-1$. The corresponding refinement mask reads

$$P(z) = \frac{1}{2n}(1 + z^n) + \frac{1}{n}\sum_{\nu=1}^{n-1} z^\nu = \frac{(1+z)(1-z^n)}{2n(1-z)},$$

and we have $P(1) = 1$. The symmetry property, $c_\nu = c_{n-\nu}$, implies that $\varphi(t) = \varphi(n-t)$ (cf. the Introduction).
Observe that, for $n = 4$, this refinement mask coincides with that of Example 4.1.
In the special case $n = 2^k$, we obtain

$$\begin{aligned} P(z) = P_k(z) \; &:= \; \frac{1}{2^{k+1}} \frac{(1+z)\,(1-z^{2^k})}{1-z} \\ &= \; \frac{1}{2^{k+1}} (1+z)^2\,(1+z^2)\,(1+z^4)\,\ldots(1+z^{2^{k-1}}). \end{aligned}$$

Let $\varphi_k(t)$ $(k \geq 1)$ be a compactly supported solution of (1.1) with the refinement mask $P_k(z)$. Then supp $\varphi_k = [0, 2^k]$. The Fourier transform $\hat{\varphi}_k$ can be given explicitly, since from

$$\prod_{k=1}^{\infty} \left( \frac{1 + e^{-iu2^l/2^k}}{2} \right) = \frac{1 - e^{-iu2^l}}{2^l iu} \quad (l \in \mathbb{N}_0)$$

it follows that

$$\hat{\varphi}_k(u) = \left( \frac{1 - e^{-iu}}{iu} \right)^2 \left( \frac{1 - e^{-2iu}}{2iu} \right) \ldots \left( \frac{1 - e^{-2^{k-1}iu}}{2^{k-1}iu} \right).$$

In time domain, $\varphi_k(t)$ can be interpreted as a convolution of characteristic functions, namely

$$\varphi_k(t) = \left( \chi_{[0,1]} \star \chi_{[0,1]} \star \chi_{[0,2]} \star \ldots \star \chi_{[0,2^{k-1}]} \right)(t).$$

Using the notion of cardinal B-spline $N_{k+1}$ of order $k+1$, defined by

$$\hat{N}_{k+1}(u) = \left(\frac{1-e^{-iu}}{iu}\right)^{k+1},$$

and the identity

$$\begin{aligned} & (1-z)^2\,(1-z^2)\ldots(1-z^{2^{k-1}}) \\ = & \left((1+z)\,(1+z+z^2+z^3)\ldots(1+z+\ldots+z^{2^{k-1}-1})\right)\,(1-z)^{k+1} \\ = & \left((1+z)^{k-1}\,(1+z^2)^{k-2}\,(1+z^4)^{k-3}\ldots(1+z^{2^{k-2}})\right)\,(1-z)^{k+1} \end{aligned}$$

it follows that

$$\hat{\varphi}_k(u) = Q_k(e^{-iu})\,\hat{N}_{k+1}(u)$$

with

$$Q_k(z) = \left(\frac{1+z}{2}\right)^{k-1}\,\left(\frac{1+z^2}{2}\right)^{k-2}\,\left(\frac{1+z^4}{2}\right)^{k-3}\ldots\left(\frac{1+z^{2^{k-2}}}{2}\right) = \sum_{n=0}^{2^k-k-1} a_n^k\,z^n.$$

Hence, with the just defined coefficients $a_n^k$, we have in time domain

$$\varphi_k(t) = \sum_{n=0}^{2^k-k-1} a_n^k\,N_{k+1}(t-n). \tag{4.7}$$

In particular, $\varphi_k \in C^{k-1}(\mathbb{R})$. The functions $f_k(s) = 2^{-k}\varphi_k(2^k s)$ can be shown to converge to a solution of (4.5). Let us mention that (4.7) can also be considered as an example for Theorem 3.6 in [2].

## 5. A NEW ALGORITHM

As known, the subdivision algorithm works well only for solutions of (1.1) with linear independent integer translates (cf. [11, 3]). Now, we want to propose a new algorithm, which is based on the fact, that we can find a factorization of $P(z)$ into polynomials

$$P(z) = \tilde{P}(z)\,Q(z) \tag{5.1}$$

with $\tilde{P}(1) = Q(1) = 1$, where the solutions of (1.1) corresponding to $\tilde{P}(z)$ can be explicitly computed as e.g. in Theorem 3.1. Remember that, if (1.1) yields a solution with linear independent (or stable) integer translates, then $P(z)$ contains a factor of the form $(1+z)^l$ with $l \geq 1$.
Assume now, that $P(z)$ factorizes as given in (5.1), and the solution $\varphi_0$ of (1.1) with the refinement mask $\tilde{P}(z)$ is explicitely known. Further, let $Q(z)$ be of the form $Q(z) = \sum_{\nu=0}^{k} d_\nu z^\nu$ $(k \geq 1)$. We define for $m \geq 1$

$$\varphi_m(t) := \sum_{\nu=0}^{k} d_\nu\,\varphi_{m-1}(t - \frac{\nu}{2^m}), \tag{5.2}$$

or in Fourier domain

$$\hat{\varphi}_m(u) := Q(e^{-iu/2^m})\,\hat{\varphi}_{m-1}(u) = \hat{\varphi}_0(u)\prod_{l=1}^{m} Q(e^{-iu/2^l}). \tag{5.3}$$

Then we have:

**Theorem 5.1** *Let the refinement mask $P(z)$ with $P(1) = 1$ be of the form* (5.1)*, and let $\varphi_0 \in C^r(\mathbb{R})$ $(r \geq 1)$ be a nonzero compactly supported solution of* (1.1) *with corresponding refinement mask $\tilde{P}(z)$. Assume that $Q(z) = \sum_{\nu=0}^{k} d_\nu\, z^\nu$ with $D := \sum_{\nu=0}^{k} |d_\nu|$ satisfying*

$$2^{r-p-1} \leq D < 2^{r-p} \tag{5.4}$$

*for some integer $p$ and $0 \leq p < r$. Then $\varphi_m$ defined in* (5.2) *converges uniformly to a solution $\varphi \in C^p(\mathbb{R})$ of* (1.1) *with corresponding $P(z)$, i.e.,*

$$\lim_{m\to\infty} \|\varphi_m - \varphi\|_\infty = 0.$$

**Proof:** Since $\varphi_0 \in C^r(\mathbb{R})$, it follows by (5.2) that $\varphi_m \in C^r(\mathbb{R})$ for $m = 1, 2, \ldots$. Observe that supp $\varphi_0 = [0, n-k]$ since $\tilde{P}(z)$ has the degree $n - k$. Then there are constants $M_l$ $(l = 0, \ldots r)$ such that

$$\|\varphi_0^{(l)}\| = \sup_{t\in\mathbb{R}} |\varphi_0^{(l)}(t)| \leq M_l \quad (l = 0, \ldots, r).$$

By (5.2) and (5.4), we easily estimate $\|\varphi_m^{(l)}\|_\infty \leq D\,\|\varphi_{m-1}^{(l)}\|_\infty$, hence

$$\|\varphi_m^{(l)}\|_\infty \leq D^m\, M_l \tag{5.5}$$

for all $m \geq 0$. Further, by $Q(1) = \sum_{\nu=0}^{k} d_\nu = 1$,

$$\varphi_m^{(l)}(t) - \varphi_{m-1}^{(l)}(t) = \sum_{\nu=0}^{k} d_\nu \left(-\varphi_{m-1}^{(l)}(t) + \varphi_{m-1}^{(l)}(t - \frac{\nu}{2^m})\right),$$

such that for $l < r$

$$\begin{aligned} |\varphi_m^{(l)}(t) - \varphi_{m-1}^{(l)}(t)| &\leq D \max_{0\leq\nu\leq k} |\varphi_{m-1}^{(l)}(t) - \varphi_{m-1}^{(l)}(t - \frac{\nu}{2^m})| \\ &\leq D\,\frac{k}{2^m}\,|\varphi_{m-1}^{(l+1)}(\xi_{m-1,l+1})| \end{aligned} \tag{5.6}$$

for some $\xi_{m-1,l+1} \in [t - \frac{k}{2^m},\, t]$. Hence, we get for $l < r$

$$\begin{aligned} |\varphi_m^{(l)}(t)| &\leq |\varphi_0^{(l)}(t)| + \sum_{\mu=1}^{m} |\varphi_\mu^{(l)}(t) - \varphi_{\mu-1}^{(l)}(t)| \\ &\leq M_l + \sum_{\mu=1}^{m} D\,\frac{k}{2^\mu}\,|\varphi_{\mu-1}^{(l+1)}(\xi_{\mu-1,l+1})|. \end{aligned}$$

By incomplete induction, we show that for $l = r,\ r-1, \ldots, p+2$ there are constants $C_l$ not depending on $m$, such that

$$\|\varphi_m^{(l)}\|_\infty \leq C_l \left(\frac{D}{2^{r-l}}\right)^m \tag{5.7}$$

is satisfied: For $l = r$ the assumption (5.7) follows by (5.5), where $C_r = M_r$. Now, supposing that (5.7) is satisfied for $l+1$ (instead of $l$) with a constant $C_{l+1}$ and $p+2 \leq l \leq r-1$, there exists a constant $C_l$ with

$$\begin{aligned}
|\varphi_m^{(l)}(t)| &= M_l + \sum_{\mu=1}^{m} D \frac{k}{2^\mu} \|\varphi_{\mu-1}^{(l+1)}\|_\infty \\
&\leq M_l + \sum_{\mu=1}^{m} D \frac{k}{2^\mu} C_{l+1} \left(\frac{D}{2^{r-l-1}}\right)^{\mu-1} \\
&= M_l + D\, C_{l+1} \frac{k}{2} \sum_{\mu=1}^{m} \left(\frac{D}{2^{r-l}}\right)^{\mu-1} \leq C_l \left(\frac{D}{2^{r-l}}\right)^m,
\end{aligned}$$

since $\frac{D}{2^{r-l}} \geq \frac{D}{2^{r-p-2}} > \frac{D}{2^{r-p-1}} \geq 1$ in view of (5.4), and (5.7) is proved. Analogously, we can show that (5.7) is also valid for $l = p+1$, if $D > 2^{r-p-1}$. We conclude in these cases that $\varphi_m^{(p)}$ is a Cauchy sequence, since by (5.6) and (5.7)

$$|\varphi_m^{(p)}(t) - \varphi_{m-1}^{(p)}(t)| \leq D \frac{k}{2^m} C_{p+1} \left(\frac{D}{2^{r-p-1}}\right)^{m-1} = \frac{kD}{2} C_{p+1} \left(\frac{D}{2^{r-p}}\right)^{m-1}$$

such that, by (5.4), $\lim_{m\to\infty} \|\varphi_m^{(p)} - \varphi_{m-1}^{(p)}\|_\infty = 0$. In case of $D = 2^{r-p-1}$ and $l = p+1$, we have

$$|\varphi_m^{(p+1)}(t)| \leq M_{p+1} + DC_{p+2}\frac{k}{2}m$$

and hence

$$|\varphi_m^{(p)}(t) - \varphi_{m-1}^{(p)}(t)| \leq D \frac{k}{2^m} \|\varphi_{m-1}^{(p+1)}\|_\infty \leq \frac{kD}{2^m} \left(M_{p+1} + D\, C_{p+2} \frac{k}{2}(m-1)\right).$$

Thus, in any case, $\varphi_m^{(p)}$ uniformly converges to a function $\varphi^{(p)}$, and therefore $\varphi_m$ uniformly converges to $\varphi$. From (1.4) and (5.3), we see that the Fourier transform of $\varphi$ has the representation $\hat\varphi(u) = \hat\varphi_0(u)\, \hat\phi(u)$, where $\phi(t)$ is the solution of (1.1) with the refinement mask $Q(z)$ normed by $\hat\phi(0) = 1$. Hence by (5.1), $\varphi(t)$ is the convolution $(\varphi_0 \star \phi)(t)$ and therefore a solution of the original two-scale difference equation (1.1) with the refinement mask $P(z)$. ∎

**Remark:** In view of $Q(1) = 1$, we always have $D \geq 1$, and for $D = 1$, we have $p = r-1 \geq 0$. The assumption $\varphi_0 \in C^r$ in Theorem 5.1 can be relaxed. It suffices to assume that $\varphi_0^{(r)}(t)$ is a piecewise continuous function. Assume that $\varphi_0 \in C^{r-1}$, and $\varphi_0^{(r)}$ is continuous up to a set of finite points $0 \leq t_1 < \ldots < t_d \leq n-k$, in which $\varphi_0^{(r)}$ can have jumps. Then, we can find an $M_r$ with $|\varphi_0^{(r)}(t)| \leq M_r$ for $t \in [0,\ n-k]$, $t \neq t_j$

$(j = 1, \ldots, d)$. Assume that $m$ satisfies $k\,2^{-m} < \min_{2\le j\le d} |t_j - t_{j-1}|$. Thus, if $t$ is in the neighborhood of $t_j$, say $t - \frac{\nu}{2^m} < t_j < t$, we have

$$\begin{aligned}
& |\varphi_{m-1}^{(r-1)}(t) - \varphi_{m-1}^{(r-1)}(t - \frac{\nu}{2^m})| \\
\le\ & |\varphi_{m-1}^{(r-1)}(t) - \varphi_{m-1}^{(r-1)}(t_j)| + |\varphi_{m-1}^{(r-1)}(t_j) - \varphi_{m-1}^{(r-1)}(t - \frac{\nu}{2^m})| \\
\le\ & (t - t_j)\,|\varphi_{m-1}^{(r)}(\xi_{j,1})| + (t_j + \frac{\nu}{2^m} - t)\,|\varphi_{m-1}^{(r)}(\xi_{j,2})|,
\end{aligned}$$

such that

$$|\varphi_{m-1}^{(r-1)}(t) - \varphi_{m-1}^{(r-1)}(t - \frac{\nu}{2^m})| \le \frac{k}{2^m}\,M_r$$

as needed in (5.6) of the proof of Theorem 5.1.

## 6. SOLUTIONS WITH NONCOMPACT SUPPORT

Up to now, we only have considered solutions of (1.1) with compact support, where $2\,P(1) = \sum_{\nu=0}^{n} c_\nu = 2^k$ $(k \in \mathbb{N})$. Now, we want to deal with the case of solutions with noncompact support and arbitrary integers $k \le 0$. We obtain:

**Theorem 6.1** *If $2\,P(1) = 2^{-k}$ with $k \in \mathbb{N}$, $k \ge 0$, then equation (1.1) possesses a unique polynomial solution of the form*

$$\varphi(t) = \sum_{\mu=0}^{k} x_\mu\, t^{k-\mu} \tag{6.1}$$

*for all $t \in \mathbb{R}$ with $x_0 = 1$.*

**Proof:** Let $\varphi$ be of the form (6.1). This function is a solution of the refinement equation (1.1), if

$$\sum_{i=0}^{k} 2^{i-k}\, x_i\, t^{k-i} = \sum_{\nu=0}^{n} c_\nu \sum_{\mu=0}^{k} x_\mu \sum_{j=0}^{k-\mu} \binom{k-\mu}{j} (-\nu)^{k-\mu-j}\, t^j.$$

Comparing the coefficients of $t^j$ with $j = k - i$, this equation is satisfied if

$$2^{i-k}\, x_i = \sum_{\nu=0}^{n} c_\nu \sum_{\mu=0}^{i} \binom{k-\mu}{k-i} (-\nu)^{i-\mu}\, x_\mu.$$

For $i = 0$, this is an identity in view of $P(1) = 2^{-k-1}$. The remaining system can be written in the form

$$(2^i - 1)\,2^{-k}\, x_i = \sum_{\mu=0}^{i-1} \binom{k-\mu}{k-i} x_\mu \sum_{\nu=1}^{n} (-\nu)^{i-\mu}\, c_\nu. \tag{6.2}$$

In view of $x_0 = 1$, it can be solved recursively for $i = 1, 2, \ldots, k$ so that the theorem is proved. $\blacksquare$

For $i = 1$, it follows that $x_1 = -k\, 2^k \sum_{\nu=1}^{n} \nu\, c_\nu$. In the case $n = 1$, we can determine the dependence of the coefficients $x_\mu$ from $k$:

**Lemma 6.2** *If $n = 1$ and $2\,P(1) = c_0 + c_1 = 2^{-k}$ with $k \in \mathbb{N}$, $k \geq 0$, then equation (1.1) has a unique solution of the form (6.1) with $x_0 = 1$ and with*

$$x_\mu = (-1)^\mu \binom{k}{\mu} \sum_{\nu=0}^{\mu} (2^k c_1)^\nu\, c_{\mu\nu}, \tag{6.3}$$

*where the coefficients $c_{\mu\nu}$ are determined recursively by $c_{00} = 1$, $c_{\mu 0} = 0$ for $\mu > 0$ and*

$$c_{\mu,\nu+1} = \frac{1}{2^\mu - 1} \sum_{k=\nu}^{\mu-1} \binom{\mu}{k} c_{k\nu} \tag{6.4}$$

*for $\mu > \nu$.*

**Proof:** Replacing (6.3) into (6.2), we obtain

$$(2^i - 1) \binom{k}{i} \sum_{\nu=0}^{i} (2^k c_1)^\nu\, c_{i\nu} = 2^k\, c_1 \sum_{\mu=0}^{i-1} \binom{k-\mu}{k-i} \binom{k}{\mu} \sum_{\nu=0}^{\mu} (2^k c_1)^\nu\, c_{\mu\nu}.$$

In view of

$$\binom{k-\mu}{k-i} \binom{k}{\mu} = \binom{k}{i} \binom{i}{\mu},$$

we find by comparison of the coefficients from $(2^k c_1)^{\nu+1}$ that $c_{\mu 0} = 0$ for $\mu > 0$ and

$$(2^i - 1) c_{i,\nu+1} = \sum_{\mu=\nu}^{i-1} \binom{i}{\mu} c_{\mu\nu},$$

i.e. (6.4). The assertion $c_{00} = 1$ follows from (6.3) and $x_0 = 1$. $\blacksquare$

As special cases of (6.4) we obtain

$$c_{\mu 1} = \frac{1}{2^\mu - 1}, \qquad c_{\mu 2} = \frac{1}{2^\mu - 1} \sum_{k=1}^{\mu-1} \binom{\mu}{k} \frac{1}{2^k - 1},$$

and the recursion formula

$$c_{\nu+1,\nu+1} = \frac{\nu + 1}{2^{\nu+1} - 1} c_{\nu\nu}.$$

Moreover, being interested in solutions of (1.1) which vanish for $t \leq 0$ and which are polynomials for $t \geq n$, we find:

**Theorem 6.3** *Let* $2P(1) = \sum_{\nu=0}^{n} c_\nu = 2^{-k}$ *with* $k \in \mathbb{N}$, $k \geq 0$, *and let*

$$\lambda = \sum_{\nu=0}^{n} |c_\nu| < 1 \tag{6.5}$$

*be satisfied. Then* (1.1) *has a unique continuous solution, which vanishes for* $t \leq 0$ *identically, and coincides with the polynomial solution in* (6.1) *for* $t \geq n$.

**Proof:** Let us introduce the operator $L$, defined by

$$L\varphi(t) := \begin{cases} \sum\limits_{\nu=0}^{m} c_\nu\, \varphi(2t-\nu) & \text{for } 0 \leq \frac{m}{2} \leq t \leq \frac{m+1}{2} \leq \frac{n}{2}, \\ \sum\limits_{\nu=0}^{m-n} c_\nu\, p(2t-\nu) + \sum\limits_{\nu=m-n+1}^{n} c_\nu\, \varphi(2t-\nu) & \text{for } \frac{n}{2} \leq \frac{m}{2} \leq t \leq \frac{m+1}{2} \leq n, \end{cases}$$

where $p(t)$ is the polynomial solution of (1.1) constructed in Theorem 6.1. First, we show the following: If $\varphi$ is a solution of (1.1) with $\varphi(t) = 0$ for $t \leq 0$ and $\varphi(t) = p(t)$ for $t \geq n$, then we have $\varphi(t) = L\varphi(t)$.
For $0 \leq \frac{m}{2} \leq t \leq \frac{m+1}{2} \leq \frac{n}{2}$ it follows that $\varphi(2t-\nu) = 0$ for $\nu > m$, implying

$$\varphi(t) = \sum_{\nu=0}^{n} c_\nu \varphi(2t-\nu) = \sum_{\nu=0}^{m} c_\nu \varphi(2t-\nu).$$

For $\frac{n}{2} \leq \frac{m}{2} \leq t \leq \frac{m+1}{2} \leq n$, we observe that $\varphi(2t-\nu) = p(2t-\nu)$ for $\nu = 0, \ldots, m-n$, such that

$$\varphi(t) = \sum_{\nu=0}^{n} c_\nu \varphi(2t-\nu) = \sum_{\nu=0}^{m-n} c_\nu\, p(2t-\nu) + \sum_{\nu=m-n+1}^{n} c_\nu\, \varphi(2t-\nu).$$

Hence, $\varphi(t) = L\varphi(t)$ in both cases.
Second, we see that the operator $L$ maps $C([0,n]) \to C([0,n])$, where $C([0,n])$ denotes the set of continuous functions $\varphi(t)$ $(0 \leq t \leq n)$ with $\varphi(0) = 0$ and $\varphi(n) = p(n)$. Taking the maximum norm, we have $\|L\varphi - L\varphi_0\| \leq \lambda\, \|\varphi - \varphi_0\|$ with $\lambda$ in (6.5) for arbitrary functions $\varphi$ and $\varphi_0$ in $C([0,n])$, so that $L$ is contractive. Hence, the assertion of the theorem follows from Banach's fixed point theorem. ∎

If we consider the function

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t < \frac{n}{2}, \\ \sum_{\nu=0}^{m-n} c_\nu\, p(2t-\nu) & \text{for } \frac{n}{2} \leq \frac{m}{2} \leq t \leq \frac{m+1}{2} \leq n, \end{cases}$$

and the linear operator $L_0$, defined by $L_0\varphi(t) = L\varphi(t) - f(t)$, we can write (1.1) in the form $\varphi(t) = L\varphi(t) = L_0\varphi(t) + f(t)$. The unique solution of this equation can be represented by Neumann's series:

$$\varphi(t) = \sum_{m=0}^{\infty} L_0^m\, f(t),$$

though $f(t) \notin C([0,n])$. If the condition (6.5) is not satisfied, then we can attain it by integration of (1.1). Afterwards, the original solution can be found by differentiation of the integrated solution, but then it may be a distribution. Obviously, by $k+1$ further differentiations, one obtains a compactly supported (distributional) solution of an equation (1.1) with the usual condition $P(1)=1$.

**Example 6.4** Let us consider the case $n=1$ with $c_0 = c_1 = 2^{-k-1}$. Then (1.1) has a solution

$$\varphi(t) = \begin{cases} 0 & \text{for } t \le 0, \\ t^{k+1} & \text{for } 0 \le t \le 1, \\ t^{k+1} - (t-1)^{k+1} & \text{for } 1 \le t, \end{cases}$$

which is normalized by $\varphi(1)=1$. The monic polynomial solution $p(t)$ reads

$$p(t) = \frac{1}{k+1} \sum_{\nu=0}^{k} \binom{k+1}{\nu} (-1)^{k-\nu}\, t^{\nu} = \sum_{\mu=0}^{k} \binom{k}{\mu} \frac{(-1)^{\mu}}{\mu+1}\, t^{k-\mu}.$$

Comparing this result with (6.3), we obtain

$$\sum_{\nu=0}^{\mu} 2^{-\nu} c_{\mu\nu} = \frac{1}{\mu+1}.$$

**Numerical computation.** Under the assumptions of Theorem 6.3, the continuous solution of (1.1) can be constructed numerically by an extension of the dyadic interpolation method (see e.g. [4]). For this purpose, we introduce the vectors

$$\tilde{\psi}(t) = (\varphi(t), \ldots, \varphi(t+n-1))^T, \quad q(t) = (p(t+n), \ldots, p(t+2n-1))^T$$

and the matrices

$$A_0 = (c_{2i-j})_{0 \le i,j \le n-1}, \quad A_1 = (c_{2i-j})_{1 \le i,j \le n},$$

which have already been used in the Introduction, and

$$B_0 = (c_{2i-j})_{\substack{0 \le i \le n-1 \\ n \le j \le 2n-1}}, \quad B_1 = (c_{2i-j})_{\substack{1 \le i \le n \\ n+1 \le j \le 2n}}.$$

Then, generalizing (1.6), equation (1.1) with $0 \le t \le 1$ can be written in the form

$$\tilde{\psi}(\frac{t}{2}) = A_0 \tilde{\psi}(t) + B_0 q(t), \quad \tilde{\psi}(\frac{t+1}{2}) = A_1 \tilde{\psi}(t) + B_1 q(t). \tag{6.6}$$

According to (6.5), the matrix $A_0$ cannot have the eigenvalue 1. Hence, we can solve the first equation in (6.6) for $t=0$ with respect to $\tilde{\psi}(0)$, and obtain

$$\tilde{\psi}(0) = (I - A_0)^{-1} B_0\, q(0).$$

This vector can be used as a start vector for our algorithm. Applying (6.6) successively, $\tilde{\psi}(t)$ can be constructed at all dyadic rationals. The corresponding linear interpolatory splines converge in view of (6.5).

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