Sparse approximation by Prony's method and AAK theory

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In signal processing and system theory, we consider the problem of sparse approximation of structured signals. Let us assume that a discrete signal $f := (f_k)_{k=0}^{\infty}$ can be represented by a linear combination of N exponentials,

$$f_k := f(k) = \sum_{j=1}^{N} a_j \, z_j^k \,, \tag{1}$$

where $a_j \in \mathbb{C} \setminus \{0\}$ and $z_j \in \mathbb{D} := \{z \in \mathbb{C} : \, 0 < |z| < 1\}$. If a suitable number of signal values $f(\ell)$, $\ell = 0, 1, \ldots, M$ with $M \geq 2N - 1$ is given, then the parameters a_j and z_j can be uniquely determined by applying Prony's method, see e.g. [6].

Our goal is now to find a new signal $\tilde{f} := (\tilde{f}_k)_{k=0}^{\infty}$ of the form

$$\tilde{f}_k := \tilde{f}(k) = \sum_{j=1}^{n} \tilde{a}_j \, \tilde{z}_j^k \tag{2}$$

with $\tilde{a}_j \in \mathbb{C} \setminus \{0\}$ and $\tilde{z}_j \in \mathbb{D}$ such that $n < N$ and $\|f - \tilde{f}\|_{\ell^2} \leq \epsilon$.

Problems of this type have been considered already in [3] and [2]. In these papers, an approach using the theory of Adamjan, Arov and Krein [1] has been employed. Furthermore the above approximation problem is strongly related to the problem of structured low rank approximation for Hankel matrices, see e.g. [4]. However, it has been still not completely understood, how to construct the new sequence $\tilde{f}$ in an optimal way.

To solve the above problem, we also employ the AAK theory and consider for the signal of the form (1) the infinite Hankel matrix

$$\mathbf{\Gamma}_f := \begin{pmatrix} f_0 & f_1 & f_2 & \ldots \\ f_1 & f_2 & f_3 & \ldots \\ f_2 & f_3 & f_4 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (f_{k+j})_{k,j=0}^{\infty}.$$

Then, it can be simply shown that $\mathbf{\Gamma}_f$ possesses rank N, and we can order the singular values $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_{N-1} > \sigma_N = \ldots = \sigma_\infty = 0$. In particular, $\mathbf{\Gamma}_f$ defines a compact operator on $\ell^2(\mathbb{N}_0)$. For the considered case, a theorem of Adamjan, Arov and Krein [1] states the following.

Theorem 1 (see [1]**).** *Let f be given as in* (1)*. Further, let $(\sigma_n, u^{(n)})$ with $u^{(n)} = (u_k^{(n)})_{k=0}^{\infty} \in \ell^2(\mathbb{N}_0)$ be a fixed singular pair of $\mathbf{\Gamma}_f$ with $\sigma_n \neq \sigma_k$ for $n \neq k$ and $\sigma_n \neq 0$. Then the series*

$$P_{u^{(n)}}(z) := \sum_{k=0}^{\infty} u_k^{(n)} \, z^k$$

has exactly n zeros $\tilde{z}_1, \ldots, \tilde{z}_n$ in $\mathbb{D}$, repeated according to their multiplicity. Moreover, if $\tilde{z}_1, \ldots, \tilde{z}_n$ are pairwise different, then there exist coefficients $\tilde{a}_1, \ldots, \tilde{a}_n \in \mathbb{C}$

such that for

$$\tilde{f} = (\tilde{f}_j)_{j=0}^{\infty} = \left(\sum_{k=1}^{n} \tilde{a}_k \tilde{z}_k^j \right)_{j=0}^{\infty}$$

we have

$$\|\mathbf{\Gamma}_f - \mathbf{\Gamma}_{\tilde{f}}\| = \sigma_n.$$

The theory behind the above theorem is presented in details in [5]. Note that due to the required structure of $\tilde{f}$ the Hankel matrix $\mathbf{\Gamma}_{\tilde{f}}$ has rank n. Therefore the theorem presents an approach for low rank Hankel approximation, standing in contrast with the approximation by usual singular value decomposition, which doesn't preserve the Hankel structure.

We want to apply this theorem to our sparse approximation problem and will answer the following questions. How ist the operator norm of the Hankel matrix $\mathbf{\Gamma}_f$ related to $\|f\|_{\ell^2}$? How to compute the singular pairs $(\sigma_n, u^{(n)})$ for $n = 0, \ldots, N-1$ numerically? How to find all zeros of the expansion $P_{u^{(n)}}(z)$ lying inside $\mathbb{D}$? How to obtain the optimal coefficients $\tilde{a}_k$?

Using the sequence $e_1 := (1, 0, 0, \ldots)^T \in \ell^2(\mathbb{N}_0)$, it follows that

$$\|f\|_{\ell^2} = \left(\sum_{j=0}^{\infty} |f_j|^2 \right)^{1/2} = \|\mathbf{\Gamma}_f e_1\|_{\ell^2} \leq \sup_{\|u\|_{\ell^2}=1} \|\mathbf{\Gamma}_f u\|_{\ell^2} = \|\mathbf{\Gamma}_f\|.$$

Therefore we have for two sequences $f, \tilde{f} \in \ell^2(\mathbb{N}_0)$ that $\|f - \tilde{f}\|_{\ell^2} \leq \|\mathbf{\Gamma}_f - \mathbf{\Gamma}_{\tilde{f}}\|$.

In order to compute the singular pairs of $\mathbf{\Gamma}_f$ we show the following theorem on the structure of singular vectors resp. con-eigenvectors of $\mathbf{\Gamma}_f$.

Theorem 2. *Let f be of the form* (1). *Then the con-eigenvectors $u^{(l)} = (u_k^{(l)})_{k=0}^{\infty}$, $l = 0, \ldots, N-1$, corresponding to the nonzero con-eigenvalues $\sigma_0 \geq \ldots \geq \sigma_{N-1} > 0$ of $\mathbf{\Gamma}_f$ are of the form*

$$u_k^{(l)} = \frac{1}{\sigma_l} \sum_{j=1}^{N} a_j P_{\overline{u}^{(l)}}(z_j) z_j^k, \qquad k \in \mathbb{N}_0,$$

where the vectors $(P_{\overline{u}^{(l)}}(z_j))_{j=1}^{N} = (\overline{P_{u^{(l)}}(\overline{z}_j)})_{j=1}^{N}$, $l = 0, \ldots, N-1$, are determined by the con-eigenvectors of the finite eigenvalue problem

$$\sigma_l (P_{u^{(l)}}(\overline{z}_j))_{j=1}^{N} = \mathbf{A}_N \mathbf{Z}_N (\overline{P_{u^{(l)}}(\overline{z}_j)})_{j=1}^{N}$$

with

$$\mathbf{A}_N := \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_N \end{pmatrix}, \quad \mathbf{Z}_N := \begin{pmatrix} \frac{1}{1-|z_1|^2} & \frac{1}{1-z_1\bar{z}_2} & \cdots & \frac{1}{1-z_1\bar{z}_N} \\ \frac{1}{1-\bar{z}_1 z_2} & \frac{1}{1-|z_2|^2} & \cdots & \frac{1}{1-z_2\bar{z}_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1-\bar{z}_1 z_N} & \frac{1}{1-\bar{z}_2 z_N} & \cdots & \frac{1}{1-|z_N|^2} \end{pmatrix}.$$

Proof. Since $\mathbf{\Gamma}_f$ is symmetric, a singular pair (σ, u) of $\mathbf{\Gamma}_f$ with $u = (u_k)_{k=0}^{\infty}$ is also a con-eigenpair satisfying $\mathbf{\Gamma}_f \overline{u} = \sigma\, u$. Denoting $P_{\overline{u}}(z) := \sum_{k=0}^{\infty} \overline{u}_k z^k$ it follows by (1) that

$$\sigma u_k = (\mathbf{\Gamma}_f \overline{u})_k = \sum_{r=0}^{\infty} f_{k+r} \overline{u}_r = \sum_{r=0}^{\infty} \sum_{j=1}^{N} a_j\, z_j^{k+r}\, \overline{u}_r = \sum_{j=1}^{N} a_j\, P_{\overline{u}}(z_j)\, z_j^k. \tag{3}$$

The assertion of the theorem is now a consequence of (3) and

$$\sigma_l P_{u^{(l)}}(z) = \sigma_l \sum_{r=0}^{\infty} u_r^{(l)} z^r = \sum_{r=0}^{\infty} \sum_{j=1}^{N} a_j P_{\overline{u}^{(l)}}(z_j) z_j^r z^r = \sum_{j=1}^{N} \frac{a_j P_{\overline{u}^{(l)}}(z_j)}{1 - z_j z}$$

for $z \in \mathbb{D}$ by inserting $z = \bar{z}_k$, $k = 1, \ldots, N$. □

From the last equality we observe that $P_{u^{(n)}}(z)$ is a rational function with a numerator being a polynomial of degree at most $N-1$, which enables to compute the zeros of $P_{u^{(n)}}$. Thus the complete algorithms reads as follows.

Algorithm for sparse approximation of exponential sums.

Input: samples f_k, $k = 0, \ldots, M$ for sufficiently large $M \geq 2N - 1$.
target approximation error ϵ

(1) Find the parameters $z_j \in \mathbb{D}$ and a_j, $j = 1, \ldots, N$ of the exponential representation of f in (1) using a Prony-like method.
(2) Solve the con-eigenproblem for the matrix $\mathbf{A}_N \mathbf{Z}_N$ and determine the largest singular value σ_n with $\sigma_n < \epsilon$.
(3) Compute the n zeros $\tilde{z}_j \in \mathbb{D}$ of the con-eigenpolynomial $P_{u^{(n)}}(z)$ of $\mathbf{\Gamma}_f$ using its rational representation.
(4) Compute the coefficients $\tilde{a}_j$ by solving the minimization problem

$$\min_{\tilde{a}_1, \ldots, \tilde{a}_n} \|f - \tilde{f}\|_{\ell^2}^2 = \min_{\tilde{a}_1, \ldots, \tilde{a}_n} \sum_{k=0}^{\infty} |f_j - \sum_{j=1}^{n} \tilde{a}_j \tilde{z}_j^k|^2.$$

Output: sequence $\tilde{f}$ of the form (2) such that $\|f - \tilde{f}\|_{\ell^2} \leq \sigma_n < \epsilon$.

References

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