Tight Frame Characterization of Multiwavelet Vector Functions in Terms of the Polyphase Matrix

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Abstract

The extension principles play an important role in characterizing and constructing of wavelet frames. The common extension principles, the unitary extension principle (UEP) or the oblique extension principle (OEP), are based on the unitarity of the modulation matrix. In this paper we state the UEP and OEP for refinable function vectors in the polyphase representation. Finally, we apply our results to directional wavelets on triangles which we have constructed in a previous work. We will show that the wavelet system generates a tight frame for $L^2(\mathbb{R}^2)$.

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Key words. Tight frames; extension principles; polyphase representation; modulation matrix; directional wavelet frames.

1 Introduction

The main tools for construction and characterization of wavelet frames are the several extension principles, the UEP and OEP as well as their generalized versions, the mixed unitary extension principle (MUEP) and the mixed oblique extension principle (MOEP). They give sufficient conditions for constructing tight and dual wavelet frames for any given refinable function which generates a multiresolution analysis (MRA). These essential methods were firstly introduced in Refs. [15] and [16] and in the fundamental work of Daubechies et al. [5] for scalar refinable functions $\phi \in L^2(\mathbb{R}^d)$, see also Ref. [3]. In the last years the results have been transferred to the case of refinable function vectors with multiplicity $r > 1$. For instance, the most general principle, the MOEP, was proven in Ref. [9] for the univariate case, i.e. for $\Phi \in (L^2(\mathbb{R}))^r$, and

recently in Ref. [8] for the multivariate case, i.e. for $\Phi \in (L^2(\mathbb{R}^d))^r$. All these extension principles derive tight or dual wavelet frames using the unitarity of the (modified) modulation matrix.
In this paper we state the UEP and OEP for refinable function vectors in the polyphase representation. A similar result for scalar refinable functions is found in Refs. [1] and [4] in connection with oversampled filter banks. Indeed, the consideration of the polyphase matrix instead of the modulation matrix is motivated by applications: the polyphase decomposition leads to computationally efficient implementations of filter banks.[17, 18] But the main advantage of polyphase representation is the possibility to create new multiwavelets by an appropriate factorization. More precisely, the modulation matrix has a particular structure, since all the information is already contained in the first column; the other columns can be derived from the first column by shifting the arguments. If we want to multiply a modulation matrix by some trigonometric polynomial matrix to create another modulation matrix, this matrix has to have a particular structure. By contrast, that is not the case with a polyphase matrix because it is unstructered. This gives the opportunity to create new multiwavelets from existing ones by multiplying the polyphase matrix by some appropriate matrix factor, and it opens the possibility of factoring a given polyphase matrix into elementary matrices (see the nicely written book Ref. [11], chapter 9, and references therein).
The paper is organized as follows. In the first section we introduce the notation of refinable vector functions. In section 2, the UEP and OEP are proven in terms of polyphase matrices. We will notice that the polyphase representation simplifies the usual proofs. Then, in section 3, we recall the well-known close connection between modulation and polyphase matrices for scalar refinable functions and extend the existing equivalence to vector quantities. Finally, we apply our results to directional wavelets on triangles constructed in Ref. [13] showing that they generate a tight frame for $L^2(\mathbb{R}^2)$.

2 General Setup

Let $\Phi = (\phi_0, \ldots, \phi_{r-1})^T$ be a vector of scaling functions $\phi_i \in L^2(\mathbb{R}^d)$, $i = 0, \ldots, r-1$, with multiplicity r that satisfies a matrix refinement equation

$$\Phi(x) = |\det A|^{1/2} \sum_{k\in\mathbb{Z}^d} M_k^0 \, \Phi(Ax - k), \quad x \in \mathbb{R}^d,$$

where $A \in \mathbb{Z}^{d\times d}$ is an expanding dilation matrix, i.e. $\lim_{j\to\infty} A^{-j} = 0$, and $M_k^0 \in \mathbb{R}^{r\times r}$ are mask coefficient matrices. Applying the Fourier transform we get the refinement equation in the Fourier domain

$$\hat{\Phi}(\omega) = H^0(\omega A^{-1}) \, \hat{\Phi}(\omega A^{-1}),$$

where the points $\omega \in \hat{\mathbb{R}}^d$ in the frequency domain are given as row vectors (in opposite to the column vectors $x \in \mathbb{R}^d$ in the time domain). Here H^0 denotes the symbol of the mask $\{M_k^0\}_{k\in\mathbb{Z}}$,

$$H^0(\omega) = \frac{1}{|\det A|^{1/2}} \sum_{k\in\mathbb{Z}^d} M_k^0 \, e^{-\mathrm{i}\omega k}.$$

The Fourier transformed function vector $\hat{\Phi} = (\hat{\phi}_0, \dots, \hat{\phi}_{r-1})^T$ is taken componentwisely by

$$\hat{\phi}_i(\omega) = \int_{\mathbb{R}^d} \phi_i(x) e^{-\mathrm{i}\omega x} \, dx, \quad i = 0, \dots, r-1.$$

Now, we are able to define $m-1$ wavelet function vectors $\Psi^l = (\psi_0^l, \dots, \psi_{r-1}^l)^T, l = 1, \dots, m-1$, by

$$\hat{\Psi}^l(\omega) = H^l(\omega A^{-1}) \, \hat{\Phi}(\omega A^{-1}), \quad \omega \in \hat{\mathbb{R}}^d, l = 1, \dots, m-1,$$

where $H^l(\omega)$ are suitable 2π-periodic matrix symbols

$$H^l(\omega) = \frac{1}{|\det A|^{1/2}} \sum_{k\in\mathbb{Z}^d} M_k^l \, e^{-\mathrm{i}\omega k} \tag{2.1}$$

of the wavelet masks $\{M_k^l\}_{k\in\mathbb{Z}}$. Let be $n = |\det A|$ and let $\Gamma = \{\gamma_0, \dots, \gamma_{n-1}\}$ be a full set of digits such that the lattice $\mathbb{Z}^d$ is partitioned into n disjoint cosets $\mathbb{Z}_s^d = \{Ak + \gamma_s : k \in \mathbb{Z}^d\}$ for $s = 0, \dots, n-1$, see Refs. [2] and [6]. The $rm \times rn$-matrix

$$\mathcal{M}(\omega) := [H^l(\omega + 2\pi\gamma A^{-1})]_{l=0,\gamma\in\Gamma}^{m-1}$$

is called *modulation matrix*. The symbols $H^l, l = 0, \dots, m-1$, in (2.1) can be splitted into n polyphase components

$$H^l(\omega) = \frac{1}{|\det A|^{1/2}} \sum_{\gamma\in\Gamma} e^{-\mathrm{i}\omega\gamma} \, H_\gamma^l(\omega A) \tag{2.2}$$

with $H_\gamma^l(\omega) = \sum_{k\in\mathbb{Z}^d} M_{Ak+\gamma}^l e^{-\mathrm{i}\omega k}$ for $\gamma \in \Gamma$. The $rm \times rn$-matrix

$$\mathcal{P}(\omega) := [H_\gamma^l(\omega)]_{l=0,\gamma\in\Gamma}^{m-1}$$

is called *polyphase matrix*. $\mathcal{P}(\omega)$ is called *unitary*, if $\overline{\mathcal{P}(\omega)}^T \mathcal{P}(\omega) = I_{rn\times rn}$, whereby $I_{rn\times rn}$ denotes the unit matrix of size $rn \times rn$. Due to the particular block structure of $\mathcal{P}(\omega)$ this property is equivalent to

$$\sum_{l=0}^{m-1} \overline{H_\gamma^l(\omega)}^T H_{\gamma'}^l(\omega) = \delta_{\gamma,\gamma'} I_{r\times r}, \quad \forall \, \gamma, \gamma' \in \Gamma \tag{2.3}$$

$$\Leftrightarrow \quad \sum_{l=1}^{m-1} \overline{H_\gamma^l(\omega)}^T H_{\gamma'}^l(\omega) = \delta_{\gamma,\gamma'} I_{r\times r} - \overline{H_\gamma^0(\omega)}^T H_{\gamma'}^0(\omega), \quad \forall \, \gamma, \gamma' \in \Gamma. \tag{2.4}$$

Remark 2.1.

1. Note that using the z-notation (as usually done in the language of filter banks) one speaks of *paraunitarity*. A matrix $\mathcal{P}(z)$ is said to be paraunitary, if it is unitary for all z on the unit circle (i.e. $z = e^{\mathrm{i}\omega}$).
2. If ϕ resp. ψ are not normalized, (almost) unitarity of $\mathcal{P}$ is given by $\overline{\mathcal{P}(\omega)}^T \mathcal{P}(\omega) = c \, I_{rn\times rn}$ for $c \in \mathbb{Z}$.

3 Polyphase Matrix and Tight Frame Property

The following theorem, the main result of our paper, shows that a unitary polyphase matrix leads to a tight multiwavelet frame.

Theorem 3.1 (UEP in polyphase representation)**.** *Let Φ be a scaling function vector that satisfies the matrix refinement equation $\hat{\Phi}(\omega) = H^0(\omega A^{-1})\ \hat{\Phi}(\omega A^{-1})$. Furthermore, let $\|\hat{\Phi}(0)\|_2^2 = 1$ and $\lim_{j\to\infty} \|\hat{\Phi}(\omega A^j)\|_2^2 = 0$ be satisfied.*
Then, if the polyphase matrix $\mathcal{P}(\omega)$ is unitary for a.e. $\omega \in \hat{\mathbb{R}}^d$, the multiwavelets $\{|\det A|^{j/2}\psi_i^l(A^j \cdot -k) : l = 1,\dots,m-1; i = 0,\dots,r-1; j \in \mathbb{Z}; k \in \mathbb{Z}^d\}$ defined by $\hat{\Psi}^l(\omega) = H^l(\omega A^{-1})\hat{\Phi}(\omega A^{-1})$ generate a tight frame for $L^2(\mathbb{R}^d)$, i.e. it exists a constant $C > 0$ with

$$C\,\|f\|_2^2 = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{l=1}^{m-1}\sum_{i=0}^{r-1} |\langle f, |\det A|^{j/2}\psi_i^l(A^j\cdot -k)\rangle|^2 \quad \forall\, f \in L^2(\mathbb{R}^d).$$

Proof. Applying Parseval's identity $\langle f, g\rangle = \frac{1}{(2\pi)^{d/2}}\langle \hat{f}, \hat{g}\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L^2(\mathbb{R}^d)$, we obtain

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{l=1}^{m-1}\sum_{i=0}^{r-1} |\langle f, |\det A|^{j/2}\psi_i^l(A^j\cdot -k)\rangle|^2 \\
&\qquad = \sum_{j,k,l,i} \Big|\frac{1}{(2\pi)^{d/2}}\langle \hat{f}, |\det A|^{-j/2}\hat{\psi}_i^l(\cdot A^{-j})e^{-\mathrm{i}\cdot A^{-j}k}\rangle\Big|^2 \\
&\qquad = \frac{1}{(2\pi)^d}\sum_{j,l,i} |\det A|^j \sum_{k\in\mathbb{Z}^d} |\langle \hat{f}(\cdot A^j)\overline{\hat{\psi}_i^l}, e^{-\mathrm{i}\cdot k}\rangle|^2 \\
&\qquad = \frac{1}{(2\pi)^d}\sum_{j,l,i} |\det A|^j \int_{\hat{\mathbb{R}}^d} |\hat{f}(\omega A^j)|^2 |\hat{\psi}_i^l(\omega)|^2\, d\omega, \qquad (3.1)
\end{aligned}$$

using in the last step again Parseval's equation $\sum_{k\in\mathbb{Z}^d} |\langle \hat{g}, e^{-\mathrm{i}\cdot k}\rangle|^2 = \|\hat{g}\|_2^2 = \int_{\hat{\mathbb{R}}^d} |\hat{g}(\omega)|^2 d\omega$ for any $g \in L^2(\mathbb{R}^d)$. In the following, we consider the sum $\sum_{l=1}^{m-1}\sum_{i=0}^{r-1} |\hat{\psi}_i^l(\omega)|^2$. With the notion $\|\hat{\Psi}^l(\omega)\|_2^2 = \sum_{i=0}^{r-1} |\hat{\psi}_i^l(\omega)|^2$ we obtain

$$\begin{aligned}
\sum_{l=1}^{m-1}\sum_{i=0}^{r-1} |\hat{\psi}_i^l(\omega)|^2 &= \sum_{l=1}^{m-1} \|\hat{\Psi}^l(\omega)\|_2^2 = \sum_{l=1}^{m-1} \|H^l(\omega A^{-1})\hat{\Phi}(\omega A^{-1})\|_2^2 \\
&= \sum_{l=1}^{m-1} \overline{\hat{\Phi}(\omega A^{-1})}^T\, \overline{H^l(\omega A^{-1})}^T H^l(\omega A^{-1})\hat{\Phi}(\omega A^{-1}).
\end{aligned}$$

We make use of the polyphase decomposition of the symbol $H^l(\omega A^{-1})$ in (2.2).

$$\begin{aligned}
\sum_{l=1}^{m-1}\sum_{i=0}^{r-1}|\hat{\psi}_i^l(\omega)|^2 &= \overline{\hat{\Phi}(\omega A^{-1})}^T \sum_{l=1}^{m-1} \overline{\left(\frac{1}{|\det A|^{1/2}}\sum_{\gamma\in\Gamma} e^{-\mathrm{i}\omega A^{-1}\gamma} H_\gamma^l(\omega)\right)}^T \\
&\qquad \left(\frac{1}{|\det A|^{1/2}}\sum_{\gamma'\in\Gamma} e^{-\mathrm{i}\omega A^{-1}\gamma'} H_{\gamma'}^l(\omega)\right)\hat{\Phi}(\omega A^{-1}) \\
&= \overline{\hat{\Phi}(\omega A^{-1})}^T \frac{1}{|\det A|}\sum_{\gamma,\gamma'\in\Gamma} e^{-\mathrm{i}\omega A^{-1}(\gamma'-\gamma)} \\
&\qquad \left(\sum_{l=1}^{m-1}\overline{H_\gamma^l(\omega)}^T H_{\gamma'}^l(\omega)\right)\hat{\Phi}(\omega A^{-1}).
\end{aligned}$$

Due to the unitarity of the polyphase matrix the expression becomes with (2.4) to

$$\begin{aligned}
\sum_{l=1}^{m-1}\sum_{i=0}^{r-1}|\hat{\psi}_i^l(\omega)|^2 &= \overline{\hat{\Phi}(\omega A^{-1})}^T \frac{1}{|\det A|}\sum_{\gamma,\gamma'\in\Gamma} e^{-\mathrm{i}\omega A^{-1}(\gamma'-\gamma)}\ . \\
&\qquad \cdot\left(\delta_{\gamma,\gamma'} I_{r\times r} - \overline{H_\gamma^0(\omega)}^T H_{\gamma'}^0(\omega)\right)\hat{\Phi}(\omega A^{-1}) \\
&= \overline{\hat{\Phi}(\omega A^{-1})}^T \frac{n}{|\det A|}\hat{\Phi}(\omega A^{-1}) \\
&\quad - \overline{\hat{\Phi}(\omega A^{-1})}^T \frac{1}{|\det A|}\sum_{\gamma,\gamma'\in\Gamma} e^{-\mathrm{i}\omega A^{-1}(\gamma'-\gamma)}\overline{H_\gamma^0(\omega)}^T H_{\gamma'}^0(\omega)\hat{\Phi}(\omega A^{-1}) \\
&= \|\hat{\Phi}(\omega A^{-1})\|_2^2 - \overline{\hat{\Phi}(\omega A^{-1})}^T\overline{H^0(\omega A^{-1})}^T H^0(\omega A^{-1})\hat{\Phi}(\omega A^{-1}) \\
&= \|\hat{\Phi}(\omega A^{-1})\|_2^2 - \overline{\hat{\Phi}(\omega)}^T\hat{\Phi}(\omega) = \|\hat{\Phi}(\omega A^{-1})\|_2^2 - \|\hat{\Phi}(\omega)\|_2^2 \\
&= \sum_{i=0}^{r-1}\left(|\hat{\phi}_i(\omega A^{-1})|^2 - |\hat{\phi}_i(\omega)|^2\right).
\end{aligned}$$

Now, putting this term into (3.1) we obtain

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{l=1}^{m-1}\sum_{i=0}^{r-1}|\langle f, |\det A|^{j/2}\psi_i^l(A^j\cdot -k)\rangle|^2 \\
&\qquad = \frac{1}{(2\pi)^d}\sum_{j\in\mathbb{Z}}|\det A|^j\int_{\hat{\mathbb{R}}^d}|\hat{f}(\omega A^j)|^2\sum_{i=0}^{r-1}\left(|\hat{\phi}_i(\omega A^{-1})|^2 - |\hat{\phi}_i(\omega)|^2\right)d\omega \\
&\qquad = \frac{1}{(2\pi)^d}\int_{\hat{\mathbb{R}}^d}|\hat{f}(\omega)|^2\sum_{j\in\mathbb{Z}}\sum_{i=0}^{r-1}\left(|\hat{\phi}_i(\omega A^{-(j+1)})|^2 - |\hat{\phi}_i(\omega A^{-j})|^2\right)d\omega.
\end{aligned}$$

For the telescope sum we get according to the assumptions to $\hat{\Phi}$ and A

$$\sum_{j\in\mathbb{Z}}\sum_{i=0}^{r-1}\left(|\hat{\phi}_i(\omega A^{-(j+1)})|^2 - |\hat{\phi}_i(\omega A^{-j})|^2\right)$$

$$\begin{aligned}
&= \lim_{j\to\infty}\sum_{i=0}^{r-1}|\hat{\phi}_i(\omega A^{-(j+1)})|^2 - \lim_{j\to-\infty}\sum_{i=0}^{r-1}|\hat{\phi}_i(\omega A^{-j})|^2\\
&= \lim_{j\to\infty}\sum_{i=0}^{r-1}|\hat{\phi}_i(\omega A^{-j})|^2 - \lim_{j\to\infty}\sum_{i=0}^{r-1}|\hat{\phi}_i(\omega A^{j})|^2\\
&= \|\hat{\Phi}(0)\|_2^2 - \lim_{j\to\infty}\|\hat{\Phi}(\omega A^j)\|_2^2 = 1.
\end{aligned}$$

Therefore, using the Plancherel formula

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{l=1}^{m-1}\sum_{i=0}^{r-1}|\langle f, |\det A|^{j/2}\psi_i^l(A^j\cdot -k)\rangle|^2$$

$$= \frac{1}{(2\pi)^d}\int_{\hat{\mathbb{R}}^d}|\hat{f}(\omega)|^2\,d\omega = \frac{1}{(2\pi)^d}\|\hat{f}\|_2^2 = \|f\|_2^2.$$

$\square$

Note, that the frame constant C is equal to one, due to our normalization. Therefore, we have even a Parseval frame.
With similar arguments we can prove the more general concept of the OEP for multivariate refinable function vectors.

Theorem 3.2 (OEP in polyphase representation)**.** *Let Φ be a scaling function vector that satisfies the matrix refinement equation $\hat{\Phi}(\omega) = H^0(\omega A^{-1})\hat{\Phi}(\omega A^{-1})$. Furthermore, suppose that $S(\omega)$ is a $r\times r$ matrix whose entries are trigonometric polynomials such that $\|S(0)\hat{\Phi}(0)\|_2^2 = 1$ and $\lim_{j\to\infty}\|S(\omega A^j)\hat{\Phi}(\omega A^j)\|_2^2 = 0$.*
If for all $\gamma,\gamma'\in\Gamma$ and for a.e. $\omega\in\hat{\mathbb{R}}^d$

$$\overline{H^0_\gamma(\omega)}^T\,\overline{S(\omega)}^T S(\omega)H^0_{\gamma'}(\omega) + \sum_{l=1}^{m-1}\overline{H^l_\gamma(\omega)}^T H^l_{\gamma'}(\omega) = \delta_{\gamma,\gamma'}\overline{S(\omega A^{-1})}^T S(\omega A^{-1}),$$

then the multiwavelets $\{|\det A|^{j/2}\psi_i^l(A^j\cdot -k) : l = 1,\ldots,m-1; i = 0,\ldots,r-1; j\in\mathbb{Z}; k\in\mathbb{Z}^d\}$ defined by $\hat{\Psi}^l(\omega) = H^l(\omega A^{-1})\hat{\Phi}(\omega A^{-1})$ generate a tight frame for $L^2(\mathbb{R}^d)$.

4 Polyphase and Modulation Representation

There is an intimate relation between the modulation matrix $\mathcal{M}(\omega)$ and the polyphase matrix $\mathcal{P}(\omega)$ because of (2.2). In case of scalar refinable functions we can express this connection by matrix multiplication.

Lemma 4.1. *Let $\mathcal{M}(\omega) := [H^l(\omega + 2\pi\tilde{\gamma}A^{-1})]_{l=0,\tilde{\gamma}\in\Gamma}^{m-1}$ be the modulation matrix and $\mathcal{P}(\omega) := [H_\gamma^l(\omega)]_{l=0,\gamma\in\Gamma}^{m-1}$ the polyphase matrix of scalar refinable functions satisfying the conditions of the general setup given above. Then, the equation*

$$\mathcal{M}(\omega) = \mathcal{P}(\omega A)\, U,$$

holds, whereby

$$U = \frac{1}{|\det A|^{1/2}} [e^{-\mathrm{i}(\omega + 2\pi\tilde{\gamma}A^{-1})\gamma}]_{\gamma,\tilde{\gamma}\in\Gamma}$$

is a unitary matrix of size $n \times n$. Therefore, $\mathcal{M}(\omega)$ is unitary if and only if $\mathcal{P}(\omega)$ is unitary for a.e. $\omega \in \hat{\mathbb{R}}^d$.

Example. The simplest and well-known refinement equation is obtained in the univariate case ($d = 1, m = 2, A = n = 2$). There, we decompose the symbols $H^l,\ l = 0, 1$, into even and odd polyphase components,

$$\mathcal{M}(\omega) = \begin{pmatrix} H^0(\omega) & H^0(\omega+\pi) \\ H^1(\omega) & H^1(\omega+\pi) \end{pmatrix} = \begin{pmatrix} H_0^0(2\omega) & H_1^0(2\omega) \\ H_0^1(2\omega) & H_1^1(2\omega) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ e^{-\mathrm{i}\omega} & -e^{-\mathrm{i}\omega} \end{pmatrix}.$$

In case of vector functions it is not possible to express the relation in terms of a unitary matrix U. Nevertheless, there exists a relation between $\mathcal{M}$ and $\mathcal{P}$, which can be described using the particular block structure of both matrices. For the proof we need the following Proposition. In order to simplify the notation we define $e_{k,j} := e^{-\mathrm{i}(\omega+2\pi\gamma_j A^{-1})\gamma_k}$ for all $k, j = 0, \ldots, n-1$.

Proposition 4.2.

1. $\sum_{j=0}^{n-1} \overline{e_{k,j}} e_{k',j} = n\, \delta_{k,k'} \quad \forall\, k, k' = 0, \ldots, n-1.$
2. *For fixed $j = 0, \ldots, n-1$ the $e_{k,j},\ k = 0, \ldots, n-1$, are linear independent.*
3. *For matrices $A_{k,k'} \in \mathbb{R}^{r\times r},\ k, k' = 0, \ldots, n-1$, holds*

$$\begin{array}{rrcll} & \sum_{k'=0}^{n-1} A_{k,k'} e_{k',j} & = & e_{k,j}\ I_{r\times r} & \forall\, j, k = 0, \ldots, n-1 \\ \Leftrightarrow & A_{k,k'} & = & \delta_{k,k'}\ I_{r\times r} & \forall\, k, k' = 0, \ldots, n-1. \end{array}$$

Proof. (i) follows from Ref. [10, Lemma 2.1]; (ii) and (iii) are consequences from (i). □

Lemma 4.3. *Let $\mathcal{M}(\omega)$ be the modulation matrix of size $rm \times rn$ and let $\mathcal{P}(\omega)$ be the polyphase matrix of refinable vector functions with multiplicity r satisfying the conditions of the general setup given above.*
Then, $\mathcal{M}(\omega)$ is unitary if and only if $\mathcal{P}(\omega)$ is unitary for a.e. $\omega \in \hat{\mathbb{R}}^d$.

Proof. Let $\mathcal{M}(\omega)$ be unitary, i.e.,

$$\sum_{l=0}^{m-1} \overline{H^l(\omega + 2\pi\tilde{\gamma}_i A^{-1})}^T H^l(\omega + 2\pi\tilde{\gamma}_j A^{-1}) = \delta_{i,j} I_{r\times r},$$

for $i, j = 0, \ldots, n-1$. Decomposing the symbols H^l into their polyphase components, we obtain with (2.3)

$$\frac{1}{|\det A|} \sum_{l=0}^{m-1} \overline{\left(\sum_{k=0}^{n-1} e_{k,i} H^l_{\gamma_k}(\omega A)\right)}^T \left(\sum_{k'=0}^{n-1} e_{k',j} H^l_{\gamma_{k'}}(\omega A)\right) = \delta_{i,j} I_{r\times r}$$

$$\Leftrightarrow \quad \frac{1}{n} \sum_{k,k'=0}^{n-1} \overline{e_{k,i}} \left(\sum_{l=0}^{m-1} \overline{H^l_{\gamma_k}(\omega A)}^T H^l_{\gamma_{k'}}(\omega A)\right) e_{k',j} = \delta_{i,j} I_{r\times r}.$$

With $A_{k,k'} := \sum_{l=0}^{m-1} \overline{H^l_{\gamma_k}(\omega A)}^T H^l_{\gamma_{k'}}(\omega A)$ and Proposition 4.2(i) we have

$$\frac{1}{n} \sum_{k=0}^{n-1} \overline{e_{k,i}} \left(\sum_{k'=0}^{n-1} A_{k,k'} e_{k',j}\right) = \frac{1}{n} \sum_{k=0}^{n-1} \overline{e_{k,i}} e_{k,j} I_{r\times r},$$

and according to Proposition 4.2(ii) and (iii) this is equivalent to $A_{k,k'} = \delta_{k,k'} I_{r\times r}$ for all $k, k' = 0, \ldots, n-1$. Because of the definition of $A_{k,k'}$ this means unitarity of $\mathcal{P}(\omega A)$ which is equivalent to the unitarity of $\mathcal{P}(\omega)$. □

Thus, the UEP in terms of the modulation matrix as commonly used is equivalent to a UEP using the polyphase representation.

5 Example: Directional Wavelets on Triangles

In Ref. [13] we have constructed non-separable directional wavelets with compact support on triangles. The tight frame property of the wavelet system was proven by arguments in the time domain. Now, with the aid of Theorem 3.1 we can show this essential property in the Fourier domain.

5.1 Haar-type Scaling Functions and Wavelets

We consider the domain $\Omega := [-1, 1]^2$ and divide it into 16 triangles with the same area, see left-hand side of Fig. 1. We want to introduce a vector of characteristic functions on these 16 triangles. Let the first scaling function ϕ_0 be a characteristic function on the triangle

$$U_0 = \mathrm{conv}\{\binom{0}{0}, \binom{1/2}{1}, \binom{0}{1}\} := \{x \in \mathbb{R}^2 : 0 \leqslant x_2 \leqslant 1,\ 0 \leqslant x_1 \leqslant \tfrac{x_2}{2}\},$$

i.e.,

$$\phi_0(x) = \phi_0(x_1, x_2) = \chi_{U_0}(x_1, x_2) = \chi_{[0,1]}(\tfrac{2x_1}{x_2}) \cdot \chi_{[0,1]}(x_2).$$

The second scaling function ϕ_1 is given by

$$\phi_1(x) = \phi_1(x_1, x_2) = \chi_{U_1}(x_1, x_2) = \chi_{[1,2]}(\tfrac{2x_1}{x_2}) \cdot \chi_{[0,1]}(x_2),$$

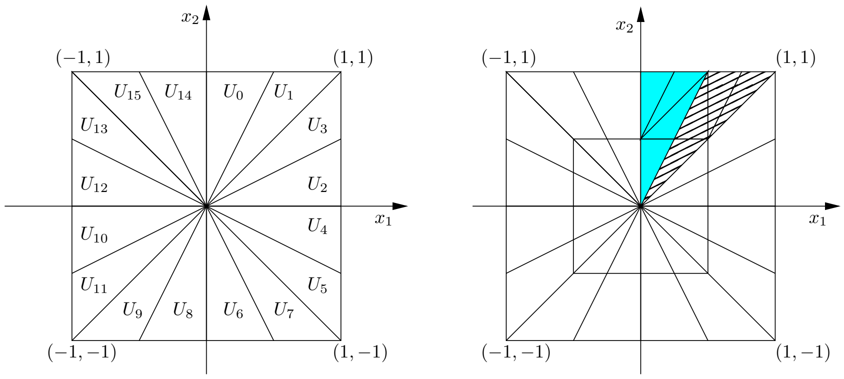

Figure 1: Supports of mother scaling functions. Left: coarsest level V_0. Right: triangle refinements.

where $U_1 = \mathrm{conv}\{\binom{0}{0}, \binom{1}{1}, \binom{1/2}{1}\}$. Let us apply the group of isometries of the square $[-1,1]^2$,

$$\mathcal{B} := \{B_i : i = 0, \ldots, 7\} = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Then, for $i = 0, \ldots, 7$ we have $U_{2i} := \{B_i^{-1}x : x \in U_0\} = B_i^{-1}U_0$ and $U_{2i+1} := \{B_i^{-1}x : x \in U_1\} = B_i^{-1}U_1$, and we define the further mother scaling functions ϕ_i by

$$\begin{aligned} \phi_{2i}(x) &:= \phi_0(B_i x) = \chi_{U_0}(B_i x) = \chi_{B_i^{-1}U_0}(x) = \chi_{U_{2i}}(x), \\ \phi_{2i+1}(x) &:= \phi_1(B_i x) = \chi_{U_1}(B_i x) = \chi_{B_i^{-1}U_1}(x) = \chi_{U_{2i+1}}(x), \qquad i = 0, \ldots, 7. \end{aligned}$$

We consider now the sequence of spaces $\{V_j\}_{j\in\mathbb{Z}}$ given by

$$V_j := \mathrm{clos}_{L^2(\mathbb{R}^2)}\mathrm{span}\{\phi_{2i,j,k}, \phi_{2i+1,j,k} : i = 0, \ldots, 7; k \in \mathbb{Z}^2\}$$

with

$$\begin{aligned} \phi_{2i,j,k}(x) &:= 2^j \phi_0(B_i(2^j x - k)), \\ \phi_{2i+1,j,k}(x) &:= 2^j \phi_1(B_i(2^j x - k)), \quad i = 0, \ldots, 7, k \in \mathbb{Z}^2. \end{aligned}$$

Note that these functions can be understood as scaling functions with composite dilations (see e.g. Refs. [7] and [14]). We have shown in Ref. [13] that $\{V_j\}_{j\in\mathbb{Z}}$ forms a generalized, stationary MRA of $L^2(\mathbb{R}^2)$, that can also be interpreted as a so-called AB-MRA with $A = 2I$ and $B \in \mathcal{B}$ as introduced in Refs. [7] and [14]. A similar approach (but only with 8 mother scaling functions and with a quincunx dilation matrix) was independently developed in Ref. [12].
For every $i = 0, \ldots, 15$, $j \in \mathbb{Z}$, and $k \in \mathbb{Z}^2$, the Haar-type scaling functions satisfy refinement equations by construction, in vector notation we have

$$\Phi(x) = 2 \sum_{k\in\mathbb{Z}^2} M_k^0 \, \Phi(2x - k), \quad x \in \mathbb{R}^2, \tag{5.1}$$

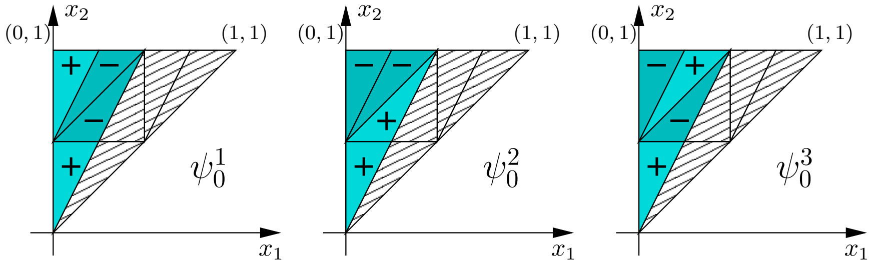

Figure 2: Directional wavelets.

where M_k^0 are 16×16-masks containing only the entries 0 or 1/2. For instance, the two-scale relation of ϕ_0 at the level $j = 0$ is given by

$$\begin{aligned} \phi_0(x) &= \phi_0(2x) + \phi_0(2x - \tbinom{0}{1}) + \phi_1(2x - \tbinom{0}{1}) + \phi_9(2x - \tbinom{1}{2}) \\ &= \frac{1}{2}\left(\phi_{0,1,0}(x) + \phi_{0,1,\binom{0}{1}}(x) + \phi_{1,1,\binom{0}{1}}(x) + \phi_{9,1,\binom{1}{2}}(x)\right), \end{aligned} \tag{5.2}$$

see right-hand side of Fig. 1. Now, we define multiwavelet vectors $\Psi^l := (\psi_i^l)_{i=0}^{15}$ by

$$\Psi^l(x) = 2 \sum_{k \in \mathbb{Z}^2} M_k^l \, \Phi(2x - k), \quad x \in \mathbb{R}^2, l = 1, 2, 3, \tag{5.3}$$

where the wavelet masks M_k^l contain entries equal to $0, 1/2$ and $-1/2$. Again we restrict to $i = 0$, where we get the wavelets

$$\psi_0^1 := \frac{1}{2}\left(\phi_{0,1,\binom{0}{0}} + \phi_{0,1,\binom{0}{1}} - \phi_{1,1,\binom{0}{1}} - \phi_{9,1,\binom{1}{2}}\right), \tag{5.4}$$

$$\psi_0^2 := \frac{1}{2}\left(\phi_{0,1,\binom{0}{0}} - \phi_{0,1,\binom{0}{1}} - \phi_{1,1,\binom{0}{1}} + \phi_{9,1,\binom{1}{2}}\right), \tag{5.5}$$

$$\psi_0^3 := \frac{1}{2}\left(\phi_{0,1,\binom{0}{0}} - \phi_{0,1,\binom{0}{1}} + \phi_{1,1,\binom{0}{1}} - \phi_{9,1,\binom{1}{2}}\right), \tag{5.6}$$

according to Fig. 2. Analogously to the scaling functions, we define the directional wavelets for every $i = 0, \ldots, 7, j \in \mathbb{Z}$ and $k \in \mathbb{Z}^2$ through

$$\psi_{2i,j,k}^l := 2^j \psi_0^l(B_i(2^j \cdot -k)), \qquad \psi_{2i+1,j,k}^l := 2^j \psi_1^l(B_i(2^j \cdot -k)), \qquad l = 1, 2, 3.$$

As already mentioned, in Ref. [13] it was shown by arguments in the spatial domain that the directional wavelet system

$$\{2^j \psi_i^l(2^j \cdot -k) : i = 0, \ldots, 15, j \in \mathbb{Z}, k \in \mathbb{Z}^2, l = 1, 2, 3\}$$

generates a Parseval frame for $L^2(\mathbb{R}^2)$. In the following we demonstrate the tight frame property in terms of the Fourier domain applying Theorem 3.1 (for $d = 2, r = 16, A = 2I, m = 4$) to our wavelet frame system.

5.2 Polyphase Matrix and Tight Frame Property

The Fourier transform of $\phi_{i,j,k} = 2^j \phi_i(2^j \cdot -k)$ can be computed easily as

$$\hat{\phi}_{i,j,k}(\omega) \quad = \quad 2^{-j} e^{-\mathrm{i}\frac{\omega}{2^j}k} \hat{\phi}_i(\frac{\omega}{2^j}).$$

Then, (5.2) and (5.4)-(5.6) leads to the Fourier domain representation

$$\begin{aligned}
\hat{\phi}_0(\omega) &= \frac{1}{4}\left(\hat{\phi}_0(\frac{\omega}{2}) + e^{-\mathrm{i}\frac{\omega}{2}\binom{0}{1}}\hat{\phi}_0(\frac{\omega}{2}) + e^{-\mathrm{i}\frac{\omega}{2}\binom{0}{1}}\hat{\phi}_1(\frac{\omega}{2}) + e^{-\mathrm{i}\frac{\omega}{2}\binom{1}{2}}\hat{\phi}_9(\frac{\omega}{2})\right), \\
\hat{\psi}_0^1(\omega) &= \frac{1}{4}\left(\hat{\phi}_0(\frac{\omega}{2}) + e^{-\mathrm{i}\frac{\omega}{2}\binom{0}{1}}\hat{\phi}_0(\frac{\omega}{2}) - e^{-\mathrm{i}\frac{\omega}{2}\binom{0}{1}}\hat{\phi}_1(\frac{\omega}{2}) - e^{-\mathrm{i}\frac{\omega}{2}\binom{1}{2}}\hat{\phi}_9(\frac{\omega}{2})\right), \\
\hat{\psi}_0^2(\omega) &= \frac{1}{4}\left(\hat{\phi}_0(\frac{\omega}{2}) - e^{-\mathrm{i}\frac{\omega}{2}\binom{0}{1}}\hat{\phi}_0(\frac{\omega}{2}) - e^{-\mathrm{i}\frac{\omega}{2}\binom{0}{1}}\hat{\phi}_1(\frac{\omega}{2}) + e^{-\mathrm{i}\frac{\omega}{2}\binom{1}{2}}\hat{\phi}_9(\frac{\omega}{2})\right), \\
\hat{\psi}_0^3(\omega) &= \frac{1}{4}\left(\hat{\phi}_0(\frac{\omega}{2}) - e^{-\mathrm{i}\frac{\omega}{2}\binom{0}{1}}\hat{\phi}_0(\frac{\omega}{2}) + e^{-\mathrm{i}\frac{\omega}{2}\binom{0}{1}}\hat{\phi}_1(\frac{\omega}{2}) - e^{-\mathrm{i}\frac{\omega}{2}\binom{1}{2}}\hat{\phi}_9(\frac{\omega}{2})\right).
\end{aligned}$$

In multiwavelet vector notation the symbol representation of (5.1) and (5.3) is given with $\Psi^0 := \Phi$ by

$$\hat{\Psi}^l(\omega) = H^l(\frac{\omega}{2})\ \hat{\Psi}^0(\frac{\omega}{2}), \quad l = 0, \ldots, 3.$$

Here the symbols H^l are finite sums of the form

$$H^l(\omega) = \frac{1}{2}\sum_{k\in\mathbb{Z}^2} M_k^l\ e^{-\mathrm{i}\omega k}$$

which can be decomposed in their polyphase components according to (2.2),

$$\begin{aligned}
H^l(\omega) &= \frac{1}{2}\left(H^l_{\binom{0}{0}} e^{-\mathrm{i}\omega\binom{0}{0}} + H^l_{\binom{1}{0}} e^{-\mathrm{i}\omega\binom{1}{0}} + H^l_{\binom{0}{1}} e^{-\mathrm{i}\omega\binom{0}{1}} + H^l_{\binom{1}{1}} e^{-\mathrm{i}\omega\binom{1}{1}}\right) \\
&= \frac{1}{2}\left(H^l_{\binom{0}{0}} + H^l_{\binom{1}{0}} e^{-\mathrm{i}\omega_1} + H^l_{\binom{0}{1}} e^{-\mathrm{i}\omega_2} + H^l_{\binom{1}{1}} e^{-\mathrm{i}(\omega_1+\omega_2)}\right),
\end{aligned}$$

since for $A = 2I$ the lattice $\mathbb{Z}^2$ can be partitioned into $n = 4$ cosets, every one represented by an integer vector from $\Gamma = \{\binom{0}{0}, \binom{1}{0}, \binom{0}{1}, \binom{1}{1}\}$. In order to obtain a simple polyphase matrix representation, we consider the translated versions of ϕ_i respectively ψ_i^l with support in $[0,1]^2$, i.e.

$$\begin{aligned}
\Psi_t^l := \big(&\psi_0^l, \ldots, \psi_3^l, \psi_4^l(\cdot - \tbinom{0}{1}), \ldots, \psi_7^l(\cdot - \tbinom{0}{1}), \\
&\psi_8^l(\cdot - \tbinom{1}{1}), \ldots, \psi_{11}^l(\cdot - \tbinom{1}{1}), \psi_{12}^l(\cdot - \tbinom{1}{0}), \ldots, \psi_{15}^l(\cdot - \tbinom{1}{0})\big)^T,
\end{aligned}$$

for $l = 0, 1, 2, 3$. Obviously, these function vectors generate the same MRA and the same wavelet system, respectively. Then the polyphase matrix corresponding to Ψ_t^l, $l =$

$0, 1, 2, 3$, is the following blockmatrix $\mathcal{P}(\omega) := [H^l_\gamma(\omega)]^3_{l=0,\gamma\in\Gamma}$, where the several blocks H^l_γ are 16×16-matrices according to the scheme

	$\gamma=\binom{0}{0}$				$\gamma=\binom{1}{0}$				$\gamma=\binom{0}{1}$				$\gamma=\binom{1}{1}$			
$l=0$	I_4	0	0	0	B_1	0	B_2	0	B_0	0	B_3	0	B_4	0	0	0
	0	B_1	0	B_2	0	B_4	0	0	0	I_4	0	0	0	B_0	0	B_3
	0	0	B_4	0	B_3	0	B_0	0	B_2	0	B_1	0	0	0	I_4	0
	0	B_3	0	B_0	0	0	0	I_4	0	0	0	B_4	0	B_2	0	B_1
$l=1$	I_4	0	0	0	$\tilde{B}_1$	0	$-B_2$	0	$\tilde{B}_0$	0	$-B_3$	0	$\tilde{B}_4$	0	0	0
	0	$\tilde{B}_1$	0	$-B_2$	0	$\tilde{B}_4$	0	0	0	I_4	0	0	0	$\tilde{B}_0$	0	$-B_3$
	0	0	$\tilde{B}_4$	0	$-B_3$	0	$\tilde{B}_0$	0	$-B_2$	0	$\tilde{B}_1$	0	0	0	I_4	0
	0	$-B_3$	0	$\tilde{B}_0$	0	0	0	I_4	0	0	0	$\tilde{B}_4$	0	$-B_2$	0	$\tilde{B}_1$
$l=2$	I_4	0	0	0	$-B_1$	0	B_2	0	$-B_0$	0	B_3	0	$-B_4$	0	0	0
	0	$-B_1$	0	B_2	0	$-B_4$	0	0	0	I_4	0	0	0	$-B_0$	0	B_3
	0	0	$-B_4$	0	B_3	0	$-B_0$	0	B_2	0	$-B_1$	0	0	0	I_4	0
	0	B_3	0	$-B_0$	0	0	0	I_4	0	0	0	$-B_4$	0	B_2	0	$-B_1$
$l=3$	I_4	0	0	0	$-\tilde{B}_1$	0	$-B_2$	0	$-\tilde{B}_0$	0	$-B_3$	0	$-\tilde{B}_4$	0	0	0
	0	$-\tilde{B}_1$	0	$-B_2$	0	$-\tilde{B}_4$	0	0	0	I_4	0	0	0	$-\tilde{B}_0$	0	$-B_3$
	0	0	$-\tilde{B}_4$	0	$-B_3$	0	$-\tilde{B}_0$	0	$-B_2$	0	$-\tilde{B}_1$	0	0	0	I_4	0
	0	$-B_3$	0	$-\tilde{B}_0$	0	0	0	I_4	0	0	0	$-\tilde{B}_4$	0	$-B_2$	0	$-\tilde{B}_1$

with

$$B_0 := \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tilde{B}_0 := \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_1 := \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tilde{B}_1 := \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_2 := \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, B_3 := \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_4 := \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \tilde{B}_4 := \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The particular block structure of the polyphase matrix leads immediately to the orthogonality of its columns. That means, $\mathcal{P}(\omega)$ is unitary. Thus the constructed wavelets form a tight frame for $L^2(\mathbb{R}^2)$ according to Theorem 3.1.

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