How many holes can locally linearly independent refinable function vectors have?

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Abstract

In this paper we consider the support properties of locally linearly independent refinable function vectors $\Phi = (\phi_1, \ldots, \phi_r)^T$. We propose an algorithm for computing the global support of the components of Φ. Further, for $\Phi = (\phi_1, \phi_2)^T$ we investigate the supports, especially the possibility of holes of refinable function vectors if local linear independence is assumed. Finally, we give some necessary conditions for local linear independence in terms of rank conditions for special matrices given by the refinement mask. But we are not able to give a final answer to the question whether a locally linearly independent function vector can have more than one hole.

1 Introduction

Let $\Phi = (\phi_1, \ldots, \phi_r)^T$, $r \in \mathbb{N}$, be a vector of compactly supported continuous functions on $\mathbb{R}$. The function vector Φ is said to be *refinable* if it satisfies a vector refinement equation

$$\Phi(x) = \sum_{k \in \mathbb{Z}} A(k)\, \Phi(2x - k), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $\{A(k)\}$ is a finitely supported sequence of real $(r \times r)$-matrices.

Refinable function vectors play a basic role in the theory of multiwavelets. In the last years the properties of refinable function vectors have been investigated very extensively. In fact, it is possible to characterize properties like approximation order and regularity of Φ and L^2-stability of the basis generated by Φ completely by means of the refinement mask $\{A(k)\}$ [1, 6, 7, 11].

We say that Φ is *L^2-stable* if there are constants $0 < A \le B < \infty$ such that for any sequences $c_1, \ldots, c_r \in l^2(\mathbb{Z})$,

$$A \sum_{\nu=1}^{r} \sum_{k \in \mathbb{Z}} |c_\nu(k)|^2 \le \left\| \sum_{\nu=1}^{r} \sum_{k \in \mathbb{Z}} c_\nu(k)\, \phi_\nu(\cdot - k) \right\|_{L^2}^2 \le B \sum_{\nu=1}^{r} \sum_{k \in \mathbb{Z}} |c_\nu(k)|^2.$$

In some applications one needs not only L^2-stability of the basis generated by Φ but other stronger conditions of linear independence. We say that Φ is *globally linearly independent* if for any sequences $c_1, \ldots, c_r$ on $\mathbb{Z}$

$$\sum_{\nu=1}^{r} \sum_{k \in \mathbb{Z}} c_\nu(k)\, \phi_\nu(\cdot - k) = 0 \quad \text{on} \quad \mathbb{R}$$

implies that $c_\nu(k) = 0$ for all $\nu = 1, \ldots, r$ and all $k \in \mathbb{Z}$ (see [8, 5]).

The following definition is even more restrictive: A function vector Φ is called to be *linearly independent on a nonempty open subset* G of $\mathbb{R}$ if for any sequences $c_1, \ldots, c_r$ on $\mathbb{Z}$

$$\sum_{\nu=1}^{r} \sum_{k \in \mathbb{Z}} c_\nu(k)\, \phi_\nu(\cdot - k) = 0 \quad \text{on} \quad G$$

implies that $c_\nu(k) = 0$ for all $k \in I_\nu(G)$, $\nu = 1, \ldots, r$, where $I_\nu(G)$ contains all $k \in \mathbb{Z}$ with $\phi_\nu(\cdot - k) \not\equiv 0$ on G. Finally, Φ is called to be *locally linearly independent* if it is linearly independent on any nonempty open subset G of $\mathbb{R}$.

Obviously, local linear independence of Φ implies global linear independence and global linear independence of Φ implies L^2-stability. It has been shown by Sun [12], that for compactly supported, refinable functions ($r = 1$) with dilation factor 2 the notions of local and global linear independence are equivalent. However, this is not longer true for function vectors [4].

For (scalar) refinable functions ϕ, local linear independence implies that ϕ has integer support, i.e., supp ϕ starts and ends with an integer, and supp ϕ does not contain holes, i.e., supp ϕ is an interval.

Now, one can ask, 'is this also true for locally linearly independent refinable function vectors?' Unfortunately this is not the case. In [10] it has been shown that a component of Φ can have a hole. However, it is not clear, whether a refinable, locally linearly independent function vector can also have components with finitely many or even infinitely many holes.

In this paper, we want to investigate support properties of locally linearly independent function vectors and consider the 'hole problem' more closely. In the second section we briefly recall a characterization of local linear independence for function vectors in terms of the mask $\{A(k)\}$. In Section 3, we present an algorithm for computing the starting points and endpoints of the support of the components ϕ_ν of Φ.

In the remaining part of the paper we restrict ourselves to the special case $\Phi = (\phi_1, \phi_2)^T$. We collect some observations on function vectors with holes in Section 4 and show that holes can only occur in special situations. In Section 5 we give necessary conditions for local linear independence in terms of rank conditions for matrices formed by the mask $\{A(k)\}$. In Section 6 we prove that the function vector Φ given in Example 4.1 is continuous and locally linearly independent. Finally, we summarize our findings in the conclusion. However, the question put in the title of this paper cannot be answered completely. We conjecture that it is not possible to have locally linearly independent function vectors with more than one hole.

2 Preliminaries

Let us start with some notations. For a compactly supported, continuous function $\phi : \mathbb{R} \to \mathbb{R}$ let $\operatorname{supp} \phi$ be the closed subset of $\mathbb{R}$, where ϕ does not vanish. Further, let the *global support* $\operatorname{gsupp} \phi$ be the smallest interval containing $\operatorname{supp} \phi$. The function ϕ is said to have a *hole* if there is an interval I which is a subset of $\operatorname{gsupp} \phi$ of Lebesgue measure greater than zero, where ϕ is identically zero. The function vector Φ is said to contain a hole if one of its components has a hole.

For a characterization of locally linearly independent function vectors we briefly recall the result of Goodman, Jia and Zhou [4]. Let Φ satisfy the refinement equation (1.1), where the mask matrices $A(k)$ are zero matrices for $k < 0$ and for $k > N$. Considering the vector

$$\mathbf{\Phi}(x) = (\Phi(x+k))_{k=0}^{N-1}$$

of length rN and the $(rN \times rN)$-block matrices

$$\mathcal{A}_0 = (A(2k-l))_{k,l=0}^{N-1}, \qquad \mathcal{A}_1 = (A(2k-l+1))_{k,l=0}^{N-1}, \tag{2.1}$$

the refinement equation can equivalently be written as

$$\mathbf{\Phi}(x/2) = \mathcal{A}_0\, \mathbf{\Phi}(x) \quad \text{and} \quad \mathbf{\Phi}((x+1)/2) = \mathcal{A}_1\, \mathbf{\Phi}(x), \qquad x \in [0,1].$$

For $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ it follows that

$$\mathbf{\Phi}\left(\frac{\epsilon_1}{2} + \cdots + \frac{\epsilon_n}{2^n} + \frac{x}{2^n}\right) = \mathcal{A}_{\epsilon_1} \cdots \mathcal{A}_{\epsilon_n}\, \mathbf{\Phi}(x), \qquad x \in [0,1].$$

Now let v_0 be a right eigenvector of $\mathcal{A}_0$ to the eigenvalue 1. This eigenvector is unique (up to multiplication with a constant) if Φ is L^2-stable (see [3]). Let V be the minimal common invariant subspace of $\{\mathcal{A}_0, \mathcal{A}_1\}$ generated by v_0. Then V contains the vectors $\mathbf{\Phi}(x)$, $x \in [0,1)$, since $\mathbf{\Phi}(0) = c\, v_0$ with some constant c and each $x \in [0,1)$ can be represented as a limit of a sequence of dyadic numbers $l/2^n$, $l \in \mathbb{Z}$, $n = 1, 2, \ldots$. Further, let $\mathcal{M}$ be an $(rN \times \dim V)$-matrix such that the columns of $\mathcal{M}$ form a basis of V. Then we have from [4]

Theorem 2.1 *Let Φ be a refinable vector of compactly supported, continuous functions satisfying* (1.1) *with $A(k) = 0$ for $k < 0$ and $k > N$. Then we have*

(1) *Φ is linearly independent on $(0,1)$ if and only if all nonzero rows of $\mathcal{M}$ are linearly independent.*

(2) *Φ is locally linearly independent if and only if for all n with $0 \le n \le 2^{rN}$ and all $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ the nonzero rows of $\mathcal{A}_{\epsilon_n} \ldots \mathcal{A}_{\epsilon_1}\mathcal{M}$ are linearly independent.*

Remark 2.2 *A similar characterization of local linear independence is possible also for L^1-solutions of vector refinement equations (1.1) and even for distributions (see [2, 13]). Some examples of locally linearly independent function vectors can be found in [4, 10].*

3 Global support of Φ

Now we want to give an algorithm for computing the global support of the components of refinable function vectors Φ from the mask. To this end let us assume that the $(r \times r)$-matrices $A(k)$ in (1.1) are of the form $A(k) = (A_{i,j}(k))_{i,j=1}^r$. We look for $\alpha_\nu,\, \beta_\nu \in \mathbb{R}$

with gsupp $\phi_\nu = [\alpha_\nu, \beta_\nu]$. Let for all pairs (i,j), $i,j = 1,\ldots r$,

$$\begin{aligned} s_{i,j} &:= \min\{k : A_{i,j}(k) \neq 0\}, \\ g_{i,j} &:= \max\{k : A_{i,j}(k) \neq 0\}. \end{aligned}$$

Observe that $s_{i,j}$, $g_{i,j}$ are integers. The numbers α_ν can be found by the following algorithm.

Algorithm 3.1

Input: $s_{i,j}$, $i,j = 1,\ldots,r$.

(1) *Let* $p := (p_1,\ldots,p_r)$ *be a vector of length* r.
For ν *from* 1 *to* r *do* $\alpha_\nu := s_{\nu,\nu}$; $p_\nu := \nu$ *enddo.*

(2) *For* ν *from* 1 *to* r *do*
  For j *from* 1 *to* r *do*
    if $s_{\nu,j} < 2\alpha_\nu - \alpha_j$ *then* $\alpha_\nu := (s_{\nu,j} + \alpha_j)/2$; $p_\nu := j$ *endif*
  enddo
enddo.

(3) *Repeat step* (2) *as long as the vector* $p = (p_1,\ldots,p_r)$ *changes.*

(4) *Form the* $(r \times r)$*-coefficient matrix* P *with*

$$P_{i,j} = \begin{cases} 1 & \text{if} \quad i = j \quad \text{and} \quad i = p(i), \\ 2 & \text{if} \quad i = j \quad \text{and} \quad i \neq p(i), \\ -1 & \text{if} \quad i \neq j \quad \text{and} \quad j = p(i), \\ 0 & \text{elsewhere}, \end{cases}$$

and the vectors $a := (\alpha_1,\ldots,\alpha_r)^T$, $s := (s_{1,p_1},\ldots,s_{r,p_r})^T$ *and solve the linear equation system* $Pa = s$.

Output: $a = (\alpha_1,\ldots,\alpha_r)^T$.

Analogously we obtain the algorithm for the endpoints β_ν:

Algorithm 3.2

Input: $g_{i,j}$, $i,j = 1,\ldots,r$.

(1) *Let* $p := (p_1,\ldots,p_r)$ *be a vector of length* r.
For ν *from* 1 *to* r *do* $\beta_\nu := g_{\nu,\nu}$; $p_\nu := \nu$ *enddo.*

(2) *For* ν *from* 1 *to* r *do*
  For j *from* 1 *to* r *do*
    if $g_{\nu,j} > 2\beta_\nu - \beta_j$ *then* $\beta_\nu := (g_{\nu,j} + \beta_j)/2$; $p_\nu := j$ *endif*
  enddo
enddo.

(3) *Repeat step* (2) *as long as the vector* $p = (p_1,\ldots,p_r)$ *changes.*

(4) *Form the* $(r \times r)$*-coefficient matrix* P *as defined in Algorithm 3.1, and the vectors* $b := (\beta_1,\ldots,\beta_r)^T$, $g := (g_{1,p_1},\ldots,g_{r,p_r})^T$ *and solve the linear equation system* $Pb = g$.

Output: $b := (\beta_1,\ldots,\beta_r)^T$.

Proof: The refinement equation (1.1) implies for each component ϕ_ν that

$$\phi_\nu(x) = \sum_{k\in\mathbb{Z}} \sum_{j=1}^{r} A_{\nu,j}(k)\, \phi_j(2x-k).$$

In particular, it follows from the local linear independence, that for all k with $A_{\nu,j}(k) \neq 0$,

$$\operatorname{gsupp} \phi_j(2\cdot -k) \subseteq \operatorname{gsupp} \phi_\nu, \qquad \nu,\, j = 1,\dots,r,$$

that is $[(\alpha_j+k)/2, (\beta_j+k)/2] \subseteq [\alpha_\nu,\, \beta_\nu]$. Using the numbers $s_{i,j}$ and $g_{i,j}$ defined above, we obtain $(\alpha_j + s_{\nu,j})/2 \geq \alpha_\nu$ and $(\beta_j + g_{\nu,j})/2 \leq \beta_\nu$, or equivalently,

$$2\alpha_\nu - \alpha_j \leq s_{\nu,j} \quad \text{and} \quad 2\beta_\nu - \beta_j \geq g_{\nu,j} \tag{3.1}$$

for all ν, $j = 1,\dots,r$. In particular, for each fixed ν at least one of the r inequalities in (3.1) for the starting points (and for the endpoints, respectively) must be an equality.

Let us look to the first algorithm computing the starting points, the second works analogously. In the first step of the algorithm we just put $\alpha_\nu := s_{\nu,\nu}$. These $s_{\nu,\nu}$ are upper bounds of the true starting points of ϕ_ν since, for $j=\nu$, (3.1) implies $\alpha_\nu \leq s_{\nu,\nu}$. Hence it is clear that, if $2\alpha_\nu - \alpha_j$ is greater than $s_{\nu,j}$ for a fixed ν and some $j \in \{1,\dots,r\}$, then α_ν must be reduced since α_j is already an upper bound for the starting point of ϕ_j. Putting now $\alpha_\nu := (s_{\nu,j} + \alpha_j)/2$ in step 2, we obtain again an upper bound of α_ν. Repeating the second step of the algorithm we obtain decreasing sequences for α_ν (being dyadic rationals, and) approaching the exact starting values. However, if the exact starting values are not dyadic rationals then they cannot be obtained by a finite number of repetitions of step 2. That's why we consider the vector p which stores for each ν an index $j = p_\nu$ for which the inequality in (3.1) is even an equality. Then step 2 must only be repeated a few times in order to find the correct vector p. Now, we can use the r equalities

$$2\alpha_\nu - \alpha_{p_\nu} = s_{\nu,p_\nu}$$

in order to compute α_ν directly. By a suitable rearranging of the equations one obtains an $(r\times r)$-coefficient matrix

$$P := \begin{pmatrix} P_1 & 0 & 0 & \dots & 0 \\ 0 & P_2 & 0 & & \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & P_\kappa & 0 \\ & & R & & D \end{pmatrix}, \tag{3.2}$$

where P_l, $l = 1,\dots,\kappa$, are circulant matrices of the form

$$\begin{pmatrix} 2 & -1 & \dots & 0 \\ 0 & 2 & -1 & \\ & & \ddots & -1 \\ -1 & 0 & \dots & 2 \end{pmatrix},$$

D is a diagonal matrix with diagonal elements 1 or 2, and R is a matrix of dimension $\dim D \times (r - \dim D)$, with one nonvanishing entry in each row at most. For example, in the case $p = (1, 2, \ldots, r)$, P is just the $(r \times r)$-identity matrix, i.e., $\dim D = r$ and the matrices P_l and R do not occur in P. For $p = (2, 3, \ldots, r, 1)$ we find $P = P_1$ and D as well as R vanish. If p contains smaller 'cycles' of the form $(p_{n_1}, \ldots, p_{n_\mu})$ with $p_{n_j} = n_{j+1}$, $j = 1, \ldots, \mu - 1$ and $p_{n_\mu} = n_1$, then each cycle corresponds to a circulant matrix P_l in P. Since the circulant matrices P_l are invertible, the equation system is uniquely solvable. □

Example 3.3 Let $r = 4$ and let the values $s_{i,j}$, $i, j = 1, 2, 3, 4$ be given by the matrix

$$(s_{i,j})_{i,j=1}^4 = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 \end{pmatrix}.$$

Algorithm 3.1 gives
step 1: $a^T = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, 1, 1)$ and $p = (1, 2, 3, 4)$
step 2: $a^T = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1/2, 3/4, 3/4, 3/8)$ and $p = (4, 1, 1, 2)$
step 3: one repetition of step 2:
$a^T = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3/16, 19/32, 19/32, 19/64)$ and $p = (4, 1, 1, 2)$
Since p did not change no further repetition of step 2 is necessary.
step 4: We obtain

$$P = \begin{pmatrix} 2 & 0 & 0 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

which can be simply changed into a matrix of the form (3.2) by rearranging the equations for the vector $a' = (\alpha_1, \alpha_4, \alpha_2, \alpha_3)^T$. The system $Pa = s$ with $s = (0, 1, 1, 0)^T$ gives $a = (1/7, 4/7, 4/7, 2/7)^T$.

Remark 3.4 *In [10] it has been shown that for locally linearly independent refinable function vectors $\Phi = (\phi_1, \ldots, \phi_r)^T$ the starting points and the endpoints of gsupp ϕ_ν, $\nu = 1, \ldots, r$, are rational numbers of the form $k + c_r$, where $k \in \mathbb{Z}$ and $c_r \in J_r$ with*

$$J_r := \left\{ \frac{k}{(2^l - 1)2^{r-l}} : l = 1, \ldots, r,\ k = 0, \ldots, (2^l - 1)2^{r-l} - 1 \right\}.$$

4 Function vectors with holes

In contrast with the scalar case, where a locally linearly independent refinable function cannot have a hole, for function vectors this need no longer to be true.

Example 4.1 Let $\Phi = (\phi_1, \phi_2)^T$ satisfy

$$\begin{aligned} \Phi(x) &= \begin{pmatrix} 1/9 & 2/9 \\ 1/3 & 1/3 \end{pmatrix} \Phi(2x) + \begin{pmatrix} 1/3 & 1/3 \\ 1 & 0 \end{pmatrix} \Phi(2x-1) + \begin{pmatrix} 2/3 & 0 \\ 1/3 & 0 \end{pmatrix} \Phi(2x-2) \\ &\quad + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Phi(2x-7). \end{aligned}$$

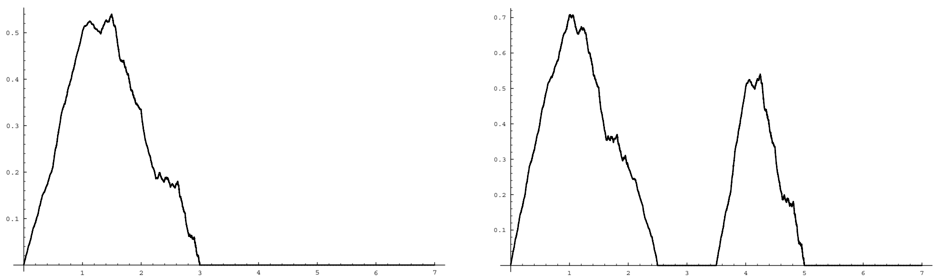

FIG. 1. Locally linearly independent function vector $\Phi = (\phi_1, \phi_2)^T$ with a hole.

Hence $\mathcal{A}_0$ and $\mathcal{A}_1$ in (2.1) are (14×14)-matrices. The function vector Φ is uniquely determined by the refinement equation (up to multiplication by a constant). Further, $\operatorname{gsupp}\phi_1 = [0, 3]$ and $\operatorname{gsupp}\phi_2 = [0, 5]$, and ϕ_2 possesses a hole of length 1, namely $\phi_2(x) = 0$ for $x \in (5/2, 7/2)$ (cf. Figure 1). As we shall show in Section 6, Φ is continuous and locally linearly independent.

Further, one can simply find function vectors Φ with infinitely many holes (but not being locally linearly independent).

Example 4.2 Let $\Phi = (\phi_1, \phi_2)^T$ with

$$\phi_1(x) = \frac{1}{2}\phi_1(2x) + \phi_1(2x-1) + \frac{1}{2}\phi_1(2x-2), \quad \phi_2(x) = \frac{1}{2}\phi_2(2x) + \phi_1(2x-4).$$

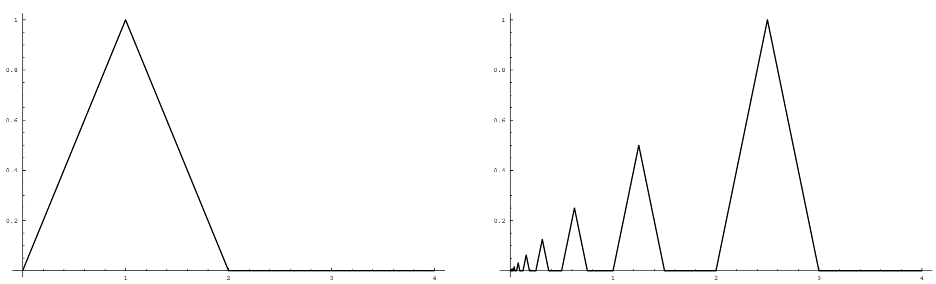

FIG. 2. Function vector $\Phi = (\phi_1, \phi_2)^T$ with infinitely many holes.

Here $\mathcal{A}_0$, $\mathcal{A}_1$ in (2.1) are (8×8)-matrices. Observe that ϕ_1 is just the hat function with $\operatorname{supp}\phi_1 = [0,2]$ and ϕ_2 is a fractal function with $\operatorname{gsupp}\phi_2 = [0, 3]$, formed by infinitely many 'hats' of support length 2^{-j}, $j = 0, 1, \ldots$, and with infinitely many holes of the form $2^{-j}(3/2, 2)$, $j = 0, 1, \ldots$ (cf. Figure 2). Of course, this function vector is not locally linearly independent, since ϕ_1 is refinable by itself (see also the proof of Theorem 4.3).

We want to consider the support properties of function vectors Φ more closely, and investigate, in which cases the components of Φ can have holes.

In the remaining part of the paper, we only investigate the case $r = 2$, i.e., $\Phi = (\phi_1, \phi_2)^T$.

Theorem 4.3 *Let $\Phi = (\phi_1, \phi_2)^T$ be a refinable, locally linearly independent vector of compactly supported, continuous functions with* $\operatorname{gsupp} \phi_\nu = [\alpha_\nu, \beta_\nu]$ *and let $l_\nu = \beta_\nu - \alpha_\nu$, $\nu = 1, 2$, be the lengths of the global supports with $l_1 \le l_2$. Suppose that Φ contains holes. Then we have*

(1) *The support lengths satisfy $l_2/2 \le l_1 < l_2$.*

(2) *There exist compactly supported, continuous functions f_1, f_2 such that $\phi_2 = f_1 + f_2$ and the vector $(\phi_1, f_1, f_2)^T$ is refinable.*

Proof: Since Φ contains holes, there exists an open interval $I = (\gamma_1, \gamma_2)$ of greatest length and a $\nu \in \{1, 2\}$ with $I \subset \operatorname{gsupp} \phi_\nu$, where ϕ_ν vanishes on I. If there are several intervals of greatest length (biggest holes) we just choose one of them. Refinability implies for $x \in I$

$$\phi_\nu(x) = 0 = \sum_k A_{\nu,1}(k)\, \phi_1(2x - k) + A_{\nu,2}(k)\, \phi_2(2x - k).$$

Since Φ is locally linearly independent, it follows that

$$\begin{array}{rcl} A_{\nu,1}(k) & = & 0 \quad \text{for} \quad \operatorname{supp} \phi_1(2 \cdot - k) \cap I \neq \emptyset, \\ A_{\nu,2}(k) & = & 0 \quad \text{for} \quad \operatorname{supp} \phi_2(2 \cdot - k) \cap I \neq \emptyset. \end{array}$$

The choice of I as the greatest interval now implies that we can replace $\operatorname{supp} \phi_\nu$ by $\operatorname{gsupp} \phi_\nu$, such that

$$\begin{array}{ll} A_{\nu,1}(k) & = 0 \quad \text{for} \quad 2\gamma_1 - \beta_1 < k < 2\gamma_2 - \alpha_1, \\ A_{\nu,2}(k) & = 0 \quad \text{for} \quad 2\gamma_1 - \beta_2 < k < 2\gamma_2 - \alpha_2. \end{array} \tag{4.1}$$

Let now $f_1 := \phi_\nu\, \chi_{[\alpha_\nu, \gamma_1]}$ and $f_2 := \phi_\nu\, \chi_{[\gamma_2, \beta_\nu]}$, where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a, b]$. Then $\phi_\nu = f_1 + f_2$ and from refinability and from (4.1) it follows that

$$\begin{array}{rcl} f_1(x) & = & \displaystyle\sum_{k \le 2\gamma_1 - \beta_1} A_{\nu,1}(k)\, \phi_1(2x - k) + \sum_{k \le 2\gamma_1 - \beta_2} A_{\nu,2}(k)\, \phi_2(2x - k), \\ f_2(x) & = & \displaystyle\sum_{k \ge 2\gamma_2 - \alpha_1} A_{\nu,1}(k)\, \phi_1(2x - k) + \sum_{k \ge 2\gamma_2 - \alpha_2} A_{\nu,2}(k)\, \phi_2(2x - k). \end{array}$$

If the hole I were in ϕ_1, then at least one of the two functions f_1, f_2 would have a global support length less than $l_1/2$ and hence would vanish since $\operatorname{gsupp} \phi_1(2 \cdot - k)$ and $\operatorname{gsupp} \phi_1(2 \cdot - k)$ have a length $\ge l_1/2$. Thus the hole must be in ϕ_2, i.e., $\phi_2 = f_1 + f_2$.

For $l_1 = l_2$ we obtain a contradiction, since, with the same argument as before, one of the two functions f_1, f_2 vanishes. Hence $l_2 > l_1$. In this case $(\phi_1, f_1, f_2)^T$ is obviously a refinable vector of continuous functions.

It remains to show that $l_2/2 > l_1$ leads to a contradiction. For $l_2/2 > l_1$, ϕ_1 must be refinable by itself, since $\operatorname{gsupp} \phi_2(2 \cdot - k)$ cannot be contained in $\operatorname{gsupp} \phi_1$ for some $k \in \mathbb{Z}$. In particular, from local linear independence we know that then $[\alpha_1, \beta_1]$ is an integer interval and that ϕ_1 has no holes. Further, since at least one of the two functions f_1, f_2

has a global support length less than $l_2/2$, it follows that this function is representable by $\phi_1(2\cdot -k)$, $k\in\mathbb{Z}$, only. Without loss of generality let

$$f_1(x) = \sum_{k\le 2\gamma_1-\beta_1} A_{2,1}(k)\,\phi_1(2x-k), \qquad x\in\mathbb{R}. \tag{4.2}$$

Considering $\mathbf{\Phi}_1 = (\phi_1(\cdot+k))_{k=\alpha_1}^{\beta_1-1}$, local linear independence implies that the space $V_1 = \operatorname{span}\{\mathbf{\Phi}_1(x) : x\in[0,1)\}$ has full dimension l_1. Further, we consider

$$\mathbf{\Phi} = \left(\mathbf{\Phi}_1^T,\ \left((f_1(\cdot+k))_{k=\lfloor\alpha_2\rfloor}^{\lceil\gamma_1\rceil-1}\right)^T,\ \left((\phi_2(\cdot+k))_{k=\lceil\gamma_1\rceil}^{\lceil\beta_2\rceil-1}\right)^T\right)^T.$$

(Here, for $x\in\mathbb{R}$, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x and $\lceil x\rceil$ denotes the smallest integer greater that or equal to x.) Now, choosing a matrix $\mathcal{M}$ of basis vectors of the space $V = \operatorname{span}\{\mathbf{\Phi}(x) : x\in[0,1)\}$, then, because of (4.2), the rows of $\mathcal{M}$ corresponding to f_1 depend on the first l_1 rows (corresponding to ϕ_1). However, not all f_1-rows can be zero rows since f_1 is not a zero function. But this contradicts the local linear independence condition by Theorem 2.1. □

Corollary 4.4 *Let $\Phi = (\phi_1,\phi_2)^T$ be a refinable, locally linearly independent vector of compactly supported, continuous functions with* $\operatorname{gsupp}\phi_\nu = [\alpha_\nu,\,\beta_\nu]$ *and $l_\nu = \beta_\nu-\alpha_\nu$, $\nu=1,2$. Suppose that $l_1\le l_2$. Then we have: If $l_1=l_2$ or $l_1<l_2/2$ then ϕ_1, ϕ_2 do not possess holes.*

Lemma 4.5 *Let $\Phi = (\phi_1,\phi_2)^T$ be a refinable, locally linearly independent vector of compactly supported, continuous functions. Then Φ has no holes that start or end with an integer.*

Proof: Suppose, Φ has a hole which ends with an integer. Choose a hole (γ_1,γ_2) of this type with biggest length. Without loss of generality assume that this hole is in ϕ_2. Then, at least in a small right neighborhood of 0, $\phi_2(\cdot+\gamma_2)$ is representable only by $\phi_1(2\cdot+\alpha_1)$ and $\phi_2(2\cdot+\alpha_2)$. Recall from [10] that the supports $\operatorname{gsupp}\phi_1 = [\alpha_1,\,\beta_1]$, $\operatorname{gsupp}\phi_2 = [\alpha_2,\,\beta_2]$ satisfy

$$\alpha_\nu = k+c_2,\quad \beta_\nu = l+c_2,\quad k,l\in\mathbb{Z},\ c_2\in\{0,1/2,1/3,2/3\}.$$

Now, if both, α_1 and α_2 are integers, then $\phi_1(x+\alpha_1)$, $\phi_2(x+\alpha_2)$, $\phi_2(x+\gamma_2)$ are linearly dependent in some suitable interval $x\in[0,\epsilon)$, $\epsilon>0$, since they can be represented by the two functions $\phi_1(2x+\alpha_1)$, $\phi_2(2x+\alpha_2)$. This is a contradiction to the local linear independence. If only one α_ν, $\nu\in\{1,2\}$ is an integer, then $\phi_\nu(x+\alpha_\nu)$ and $\phi_2(x+\gamma_2)$ are representable only by $\phi_\nu(2x+\alpha_\nu)$ in some interval $x\in[0,\epsilon)$ as before and we again obtain a contradiction. If neither α_1 nor α_2 are integers, then $\phi_2(x+\gamma_2)$ cannot be represented by integer translates of $\phi_\nu(2x)$, $\nu=1,2$, contradicting the refinability.

Analogously, the contradiction follows for holes starting with an integer. □

Let us call a hole (γ_1,γ_2) in Φ *biggest hole* if there is no other hole in Φ of double size of the form $(2\gamma_1+k,\,2\gamma_2+k)$ with some $k\in\mathbb{Z}$.

Lemma 4.6 *Let $\Phi = (\phi_1,\phi_2)^T$ be a refinable, locally linearly independent vector of compactly supported, continuous functions. Then there is at most one biggest hole in Φ.*

Proof: Assume that Φ has two biggest holes. Let again l_1, l_2 denote the lengths of the global supports of ϕ_1, ϕ_2 and suppose that $l_1 < l_2$. Then ϕ_1 cannot have a biggest hole by Theorem 4.3. Hence the two holes must be in ϕ_2 and we get a partition $\phi_2 = f_1 + f_2 + f_3$ analogously as in the proof of Theorem 4.3 such that $(\operatorname{gsupp} f_1) \cup (\operatorname{gsupp} f_2) \cup (\operatorname{gsupp} f_3) \subset \operatorname{gsupp} \phi_2$. Further, by refinability, each function f_1, f_2, f_3 can be represented by $\phi_1(2 \cdot -k)$, $\phi_2(2 \cdot -k)$, $k \in \mathbb{Z}$. Moreover, at least one of the three functions f_1, f_2, f_3 must contain a translate of $\phi_2(2\cdot)$, otherwise at least two of the functions f_1, f_2, f_3 would be linearly dependent in a suitable interval inside the starting intervals, since $\phi_1(2 \cdot -k)$ either starts at $\mathbb{Z} + \alpha_1/2$ or at $\mathbb{Z} + (\alpha_1 + 1)/2$ (depending on whether k is even or odd). Hence $\operatorname{gsupp} \phi_2 > (\operatorname{gsupp} \phi_2)/2 + 2(\operatorname{gsupp} \phi_1)/2$. But this contradicts Corollary 4.4. $\square$

Remark 4.7 *All results in this section can be generalized to $r > 2$ and to L^1-integrable functions, if the characterization of local linear independence in [2] is used.*

5 Rank conditions for matrices formed by the refinement mask

We again restrict ourselves to the case that $\Phi = (\phi_1, \phi_2)^T$ is a vector of compactly supported, continuous functions satisfying the refinement equation (1.1) with $A(k) = 0$ for $k < 0$ and $k > N$.

Let us consider the matrices $\mathcal{A}_0$ and $\mathcal{A}_1$ in (2.1) and the minimal common invariant subspace V of $\{\mathcal{A}_0, \mathcal{A}_1\}$ generated by v_0 as defined in Section 2. Recall that V contains $\mathbf{\Phi}(x)$, $x \in [0, 1)$. Let $\mathcal{M}$ be an $(rN \times \dim V)$-matrix such that the columns of $\mathcal{M}$ form a basis of V. Now delete all components in the vector $\mathbf{\Phi} = (\Phi(x+k))_{k=0}^{N-1}$ corresponding to zero rows in $\mathcal{M}$ in order to get $\widetilde{\mathbf{\Phi}}$. Further, delete the corresponding rows and columns in the matrices $\mathcal{A}_0$ and $\mathcal{A}_1$ in (2.1) in order to obtain $\widetilde{\mathcal{A}}_0$ and $\widetilde{\mathcal{A}}_1$ with

$$\widetilde{\mathbf{\Phi}}(x/2) = \widetilde{\mathcal{A}}_0 \widetilde{\mathbf{\Phi}}(x), \qquad \widetilde{\mathbf{\Phi}}((x+1)/2) = \widetilde{\mathcal{A}}_1 \widetilde{\mathbf{\Phi}}(x), \qquad x \in [0,1]. \tag{5.1}$$

Deleting the zero rows and the corresponding columns in $\mathcal{M}$ we obtain $\widetilde{\mathcal{M}}$.

Example 5.1 Let us consider Example 4.1. Here $\mathbf{\Phi}$ is a vector of length 14 and $V = \operatorname{span}\{\Phi(x+k)_{k=0}^{6} : x \in [0,1)\}$. Since $\operatorname{supp} \phi_1 = [0, 3]$ and $\operatorname{supp} \phi_2 \subset [0, 5]$, it follows that the rows of $\mathcal{M}$ corresponding to $\phi_1(x+j)$, $j = 3, 4, 5, 6$, and $\phi_2(x+j)$, $j = 5, 6$ are zero rows. Indeed, these are all zero rows of $\mathcal{M}$, i.e., V has dimension 8. We delete these components of $\mathbf{\Phi}(x)$ and obtain

$$\widetilde{\mathbf{\Phi}}(x) = (\phi_1(x), \phi_2(x), \phi_1(x+1), \phi_2(x+1), \phi_1(x+2), \phi_2(x+2), \phi_2(x+3), \phi_2(x+4))^T$$

as well as

$$9\widetilde{\mathcal{A}}_0 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 3 & 3 & 1 & 2 & 0 & 0 \\ 3 & 0 & 9 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad 9\widetilde{\mathcal{A}}_1 = \begin{pmatrix} 3 & 3 & 1 & 2 & 0 & 0 & 0 & 0 \\ 9 & 0 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 3 & 3 & 2 & 0 \\ 0 & 0 & 3 & 0 & 9 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

Let us call a row of $\widetilde{\mathcal{A}}_0$ (resp. $\widetilde{\mathcal{A}}_1$) ϕ_1-row if it corresponds to an ϕ_1-entry in $\widetilde{\mathbf{\Phi}}$ and ϕ_2-row if it correspond to an ϕ_2-entry.

Let n be the length of the new vector $\widetilde{\mathbf{\Phi}}$ and hence $\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1$ are $(n \times n)$-matrices. If Φ is a locally linearly independent vector then Theorem 2.1 implies that $\widetilde{\mathcal{M}}$ is an invertible $(n \times n)$-matrix.

Deleting the first ϕ_1-row and the first ϕ_2-row and the corresponding columns in $\widetilde{\mathcal{A}}_0$, we obtain a new matrix $\mathcal{B}$ of dimension $(n-2)\times(n-2)$. The same matrix $\mathcal{B}$ is obtained, if we delete the last ϕ_1-row and the last ϕ_2-row and corresponding columns in $\widetilde{\mathcal{A}}_1$. Further, the structure of $\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1$ implies that

$$\operatorname{spec} \widetilde{\mathcal{A}}_0 = \operatorname{spec} J_0 \cup \operatorname{spec} \mathcal{B}, \qquad \operatorname{spec} \widetilde{\mathcal{A}}_1 = \operatorname{spec} J_1 \cup \operatorname{spec} \mathcal{B},$$

where J_0 (resp. J_1) is a 2×2-matrix containing the entries of $\widetilde{\mathcal{A}}_0$ (resp. $\widetilde{\mathcal{A}}_1$) being at the same time in the first ϕ_1- or ϕ_2-row (resp. last ϕ_1- or ϕ_2-row) and in the first ϕ_1- or ϕ_2-column (resp. last ϕ_1- or ϕ_2-column). (Here $\operatorname{spec} A$ denotes the set of eigenvalues of a matrix A.)

Example 5.2 For $\Phi = (\phi_1, \phi_2)^T$ in Example 5.1 we obtain the matrix $\mathcal{B}$ after deleting the first and second row and corresponding columns in $\widetilde{\mathcal{A}}_0$ or by deleting the 5th and 8th row and corresponding columns in $\widetilde{\mathcal{A}}_1$. Hence,

$$\mathcal{B} = \frac{1}{9}\begin{pmatrix} 3 & 3 & 1 & 2 & 0 & 0 \\ 9 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 6 & 0 & 3 & 2 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_0 = \frac{1}{9}\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}, \quad J_1 = \frac{1}{9}\begin{pmatrix} 0 & 3 \\ 9 & 0 \end{pmatrix}$$

where J_1 and J_2 are invertible.

We obtain

Theorem 5.3 *Let $\Phi = (\phi_1, \phi_2)^T$ be a refinable, locally linearly independent vector of compactly supported, continuous functions. Further, let $\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1$ and $\mathcal{B}$ be given as above. Then we have*

(1) $\operatorname{rank}(J_0) \geq 1$ *and* $\operatorname{rank}(J_1) \geq 1$,
(2) $\operatorname{rank}(\mathcal{B}) \geq n-3$,
(3) $\operatorname{rank}(\widetilde{\mathcal{A}}_0) \geq n-2$ *and* $\operatorname{rank}(\widetilde{\mathcal{A}}_1) \geq n-2$,
(4) $|\operatorname{rank}(\widetilde{\mathcal{A}}_0) - \operatorname{rank}(\widetilde{\mathcal{A}}_1)| \leq 1$.

Proof: (1) First observe that J_0 and J_1 at least have rank 1, otherwise a component of $\widetilde{\mathbf{\Phi}}(x)$, $x \in [0,1)$ would completely vanish, contradicting the definition of $\widetilde{\mathbf{\Phi}}$.

Let $\operatorname{gsupp} \phi_1 = [\alpha_1, \beta_1]$ and $\operatorname{gsupp} \phi_2 = [\alpha_2, \beta_2]$. Then, one simple eigenvalue zero in J_0 implies that $\alpha_1 \in \mathbb{Z}$, $\alpha_2 \in \mathbb{Z} + 1/2$ or vice versa. If J_0 has two eigenvalues 0 then the geometric multiplicity of 0 must be 1 and we obtain $\alpha_1 \in \mathbb{Z} + 1/3$, $\alpha_2 \in \mathbb{Z} + 2/3$ or vice versa. Analogously, a corresponding behavior of J_1 implies $\beta_1 \in \mathbb{Z} + 1/2$, $\beta_2 \in \mathbb{Z}$ or vice versa, and $\beta_1 \in \mathbb{Z} + 2/3$, $\beta_2 \in \mathbb{Z} + 1/3$ or vice versa, respectively.

(2) If the matrix $\mathcal{B}$ possesses the eigenvalue zero, then both, $\widetilde{\mathcal{A}}_0$ and $\widetilde{\mathcal{A}}_1$ possess the eigenvalue zero. Hence, $\widetilde{\mathcal{A}}_0\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{A}}_1\widetilde{\mathcal{M}}$ are not invertible, while $\widetilde{\mathcal{M}}$ is an invertible matrix. Thus, by Theorem 2.1, $\widetilde{\mathcal{A}}_0$ and $\widetilde{\mathcal{A}}_1$ have a zero row, but being not the first or last ϕ_1- or ϕ_2-row. Hence, also $\mathcal{B}$ has a zero row and, by construction, if $\widetilde{\mathcal{A}}_0$ has the zero row in the l-th ϕ_i-row, $i \in \{1,2\}$, then $\widetilde{\mathcal{A}}_1$ must have a zero row in the $(l-1)$-th ϕ_i-row. This means by (5.1), the two zero rows imply a hole in Φ containing the interval $(k-1/2, k+1/2)$, for some $k \in \mathbb{Z}$. This hole must be a biggest hole. If $\mathcal{B}$ has the eigenvalue zero with geometric multiplicity greater than 1, then with the same arguments one obtains a second biggest hole in Φ. But this contradicts the local linear independence by Lemma 4.6. Hence $\text{rank}(\mathcal{B}) \geq n-3$.

(3) The above considerations directly imply that $\text{rank}(\widetilde{\mathcal{A}}_0) \geq n-2$ and $\text{rank}(\widetilde{\mathcal{A}}_1) \geq n-2$.

(4) Now, if $\widetilde{\mathcal{A}}_0$ has rank $n-2$, then $\mathcal{B}$ has rank $n-3$ and hence $\widetilde{\mathcal{A}}_1$ can have rank $n-1$ at most. Analogously, $\text{rank}(\widetilde{\mathcal{A}}_1) = n-2$ implies $\text{rank}(\widetilde{\mathcal{A}}_0) \leq n-1$. □

From Theorem 5.3 it follows that we have to investigate the following five cases:

(1) $\text{rank}(\widetilde{\mathcal{A}}_0) = \text{rank}(\widetilde{\mathcal{A}}_1) = n$,

(2) $\text{rank}(\widetilde{\mathcal{A}}_0) = \text{rank}(\widetilde{\mathcal{A}}_1) = n-1$,

(3) $\text{rank}(\widetilde{\mathcal{A}}_0) = \text{rank}(\widetilde{\mathcal{A}}_1) = n-2$,

(4) $\text{rank}(\widetilde{\mathcal{A}}_0) = n-1$, $\text{rank}(\widetilde{\mathcal{A}}_1) = n$,

(5) $\text{rank}(\widetilde{\mathcal{A}}_0) = n-1$, $\text{rank}(\widetilde{\mathcal{A}}_1) = n-2$.

All further cases can be reduced to one of the above. However, some of these cases may contradict the local linear independence assumption for Φ.

Considering the first two cases, we obtain a partial answer to the question of whether the support of ϕ_i, $i = 1,2$, can have holes. Moreover, we obtain sufficient conditions for the local linear independence of Φ in terms of rank conditions for $\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1$.

For the first case we obtain:

Theorem 5.4 *Let $\Phi = (\phi_1, \phi_2)^T$ be a refinable vector of compactly supported, continuous functions. Let the space $\widetilde{V} = span\{\widetilde{\mathbf{\Phi}}(x) : x \in [0,1)\}$ have full dimension, i.e. $\widetilde{\mathcal{M}}$, formed by basis vectors of $\widetilde{V}$ is an invertible $(n \times n)$-matrix. Let $\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1$ be given as above. Then* $\text{rank}(\widetilde{\mathcal{A}}_0) = \text{rank}(\widetilde{\mathcal{A}}_1) = n$ *implies that Φ is locally linearly independent and has no holes.*

Proof: The assertion on local linear independence is already proved in [4], Theorem 3.2. Since $\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1$ are invertible, the matrix $\widetilde{\mathcal{A}}_{\epsilon_1} \cdots \widetilde{\mathcal{A}}_{\epsilon_n}\widetilde{\mathcal{M}}$ never has a zero row, hence from

$$\widetilde{\mathbf{\Phi}}\left(\frac{\epsilon_1}{2} + \cdots + \frac{\epsilon_n}{2^n} + \frac{x}{2^n}\right) = \widetilde{\mathcal{A}}_{\epsilon_1} \cdots \widetilde{\mathcal{A}}_{\epsilon_n}\,\widetilde{\mathbf{\Phi}}(x), \qquad x \in [0,1), \tag{5.2}$$

it follows that there is no dyadic interval where ϕ_1 or ϕ_2 vanishes. Thus Φ has no holes. □

For the second case we find

Theorem 5.5 *Let $\Phi = (\phi_1, \phi_2)^T$ be a refinable vector of compactly supported, continuous functions. Let the space $\widetilde{V} = span\{\widetilde{\mathbf{\Phi}}(x) : x \in [0,1)\}$ have full dimension, i.e. $\widetilde{\mathcal{M}}$,* *formed by basis vectors of* $\widetilde{V}$ *is an invertible* $(n \times n)$*-matrix. Let* $\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1$ *and* $\mathcal{B}$ *be given as above. Further, let* $\mathrm{rank}(\widetilde{\mathcal{A}}_0) = \mathrm{rank}(\widetilde{\mathcal{A}}_1) = n-1$ *and each of these matrices has one zero row. Then we have*

(1) *If* $\mathrm{rank}(\mathcal{B}) = n-2$ *and the four matrices* $\widetilde{\mathcal{A}}_0\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_0\widetilde{\mathcal{A}}_1$, $\widetilde{\mathcal{A}}_1\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1\widetilde{\mathcal{A}}_1$ *have rank* $n-1$*, then* Φ *is locally linearly independent and has no holes.*

(2) *If* $\mathrm{rank}(\mathcal{B}) = n-3$ *and the four matrices* $\widetilde{\mathcal{A}}_0\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_0\widetilde{\mathcal{A}}_1$, $\widetilde{\mathcal{A}}_1\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1\widetilde{\mathcal{A}}_1$ *have rank* $n-1$*, then* Φ *is locally linearly independent and has one hole of the form* $(k-1/2,\, k+1/2)$ *for some* $k \in \mathbb{Z}$*.*

Proof: (1) We consider the first case. Since $\mathrm{rank}(\mathcal{B}) = n-2$, it follows that $\mathcal{B}$ is invertible and the zero row of $\widetilde{\mathcal{A}}_0$ must be the first ϕ_1-row or the first ϕ_2-row. Analogously, the zero row of $\widetilde{\mathcal{A}}_1$ must be the last ϕ_1- or ϕ_2-row. Since $\mathrm{rank}(\widetilde{\mathcal{A}}_0\widetilde{\mathcal{A}}_0) = \mathrm{rank}(\widetilde{\mathcal{A}}_1\widetilde{\mathcal{A}}_1) = n-1$, it follows that J_0 and J_1 only have a simple eigenvalue zero and the assumptions (1) of the theorem imply that all matrix products $\widetilde{\mathcal{A}}_{\epsilon_1} \cdots \widetilde{\mathcal{A}}_{\epsilon_n}\widetilde{\mathcal{M}}$, $n \in \mathbb{N}$, have rank $n-1$ and one zero row, namely the same as $\widetilde{\mathcal{A}}_0$ if $\epsilon_1 = 0$ and the same as $\widetilde{\mathcal{A}}_1$ if $\epsilon_1 = 1$. The assumption on $\widetilde{V}$ in the theorem already ensures that Φ is linearly independent on $(0,1)$. Now the above observations also imply that, by Theorem 2.1, Φ is locally linearly independent.

The zero row in $\widetilde{\mathcal{A}}_0$ implies that the support of one component of Φ starts with an integer and the support of the other with a half integer. Considering the zero row in $\widetilde{\mathcal{A}}_1$ we also find that the support of one component ends with an integer and the support of the other with a half integer. In particular, from (5.2) it follows that Φ cannot have holes.
(2) We consider the second case. Since $\mathrm{rank}(\mathcal{B}) = n-3$, it follows that $\mathcal{B}$ possesses the eigenvalue zero and the zero rows of $\widetilde{\mathcal{A}}_0$ and $\widetilde{\mathcal{A}}_1$ are not the first or the last ϕ_1- or ϕ_2-rows. Moreover, as shown in the proof of Theorem 5.3, if the l-th ϕ_i-row, $i \in \{1,2\}$, of $\widetilde{\mathcal{A}}_0$ is a zero row then the $(l-1)$-th ϕ_i-row of $\widetilde{\mathcal{A}}_1$ is also a zero row, and this implies by (5.1) a hole of the form $(k-1/2,\, k+1/2)$ for some $k \in \mathbb{Z}$ in ϕ_i. Further, the rank conditions (2) of the theorem imply that all matrix products $\widetilde{\mathcal{A}}_{\epsilon_1} \cdots \widetilde{\mathcal{A}}_{\epsilon_n}\widetilde{\mathcal{M}}$, $n \in \mathbb{N}$, have rank $n-1$ and either a zero row in the l-th or in the $(l-1)$-th row. Thus, by Theorem 2.1, Φ is locally linearly independent and has only one hole. □

Remark 5.6 *Example 4.1 satisfies the assumptions of Theorem 5.5 (2). An example satisfying Theorem 5.5 (1) can be found in [10].*

Observe that the case (2) is not completely settled by Theorem 5.5 since for $\mathrm{rank}(\widetilde{\mathcal{A}}_0) = \mathrm{rank}(\widetilde{\mathcal{A}}_1) = n-1$ some of the four matrices $\widetilde{\mathcal{A}}_0\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_0\widetilde{\mathcal{A}}_1$, $\widetilde{\mathcal{A}}_1\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1\widetilde{\mathcal{A}}_1$ can also have rank $n-2$. Indeed, there exist locally linearly independent function vectors, where $\mathrm{rank}(\widetilde{\mathcal{A}}_0) = \mathrm{rank}(\widetilde{\mathcal{A}}_1) = n-1$ and $\mathrm{rank}(\widetilde{\mathcal{A}}_0\widetilde{\mathcal{A}}_0) = \mathrm{rank}(\widetilde{\mathcal{A}}_1\widetilde{\mathcal{A}}_1) = n-2$, see [10]. The remaining cases are more complicated to handle and we cannot give a final answer to the question of whether a locally linearly independent refinable vector Φ can have more than one hole.

6 Proof of the example

In this section we want to verify the assertion that the function vector Φ given by the refinement mask in Example 4.1 is continuous and locally linearly independent. Let us first prove that Φ is continuous. To this end we use the following observation by Jia, Riemenschneider and Zhou [9]:

Let $\{A(k)\}_{k=0}^N$ be a real refinement mask satisfying the following properties:

(1) $\frac{1}{2}\sum_{k=0}^N A(k)$ has one eigenvalue 1 and all further eigenvalues are inside the unit circle.

(2) The matrices $\mathcal{A}_0$ and $\mathcal{A}_1$ both have the simple eigenvalue 1 and there is a vector $e_1 \in \mathbb{R}^{Nr}$ with $e_1^T\,\mathcal{A}_0 = e_1^T\,\mathcal{A}_1 = e_1^T$.

(3) Considering the space $U = \{u \in \mathbb{R}^{rN} : e_1^T u = 0\}$ the joint spectral radius of $\mathcal{A}_0|_U$ and $\mathcal{A}_1|_U$ satisfies $\rho(\mathcal{A}_0|_U\,\mathcal{A}_0|_U) < 1$.

Then the subdivision scheme associated with $\{A(k)\}_{k=0}^N$ converges in the maximum norm, and hence the solution vector Φ of the refinement equation is continuous.
Here the joint spectral radius satisfies for any matrix norm

$$\rho(\mathcal{A}_0|_U\,\mathcal{A}_0|_U) = \inf_{n\geq 1}(\max\{\|\mathcal{A}_{\epsilon_1}|_U\cdots\mathcal{A}_{\epsilon_n}|_U\| : \epsilon_i \in \{0,1\},\, i=1,\ldots,n\})^{1/n}.$$

For our example we find:

1) $\frac{1}{2}\sum_{k=0}^{7} A(k) = \begin{pmatrix} 5/9 & 5/18 \\ 4/3 & 1/6 \end{pmatrix}$ possesses the eigenvalues 1 and $-5/18$.

2) The matrices $\mathcal{A}_0$ and $\mathcal{A}_1$ both have the simple eigenvalue 1 with the left eigenvector $e_1^T = (3,1,3,1,3,1,3,1,3,1,3,1,3,1)$.

3) The space $U = \{u \in \mathbb{R}^{14} : e_1^T\,u = 0\}$ has dimension 13 and we find the orthonormal basis of U:

$$\begin{aligned}
u_1 &= 28^{-1/2}\,(4,0,0,0,-3,-1,0,0,0,-1,0,0,0,-1)^T,\\
u_2 &= 110^{-1/2}\,(0,0,0,0,-3,-1,0,0,0,0,0,0,0,10)^T,\\
u_3 &= 130^{-1/2}\,(-3,0,0,0,-3,-1,-3,0,0,-1,0,0,10,-1)^T,\\
u_4 &= 132^{-1/2}\,(0,0,0,0,-3,-1,0,0,0,11,0,0,0,-1)^T,\\
u_5 &= 70^{-1/2}\,(-3,0,0,0,-3,-1,7,0,0,-1,0,0,0,-1)^T,\\
u_6 &= 208^{-1/2}\,(-3,0,0,0,-3,-1,-3,0,0,-1,13,0,-3,-1)^T,\\
u_7 &= 3540^{-1/2}\,(-3,-1,-3,0,-3,-1,-3,59,0,-1,-3,-1,-3,-1)^T,\\
u_8 &= 3660^{-1/2}\,(-3,-1,-3,60,-3,-1,-3,-1,0,-1,-3,-1,-3,-1)^T,\\
u_9 &= 2352^{-1/2}\,(-3,48,0,0,-3,-1,-3,0,0,-1,-3,0,-3,-1)^T,\\
u_{10} &= 3422^{-1/2}\,(-3,-1,-3,0,-3,-1,-3,0,0,-1,-3,58,-3,-1)^T,\\
u_{11} &= 4270^{-1/2}\,(-9,-3,-9,-3,-9,-3,-9,-3,61,-3,-9,-3,-9,-3)^T,\\
u_{12} &= 10^{-1/2}\,(0,0,0,0,1,-3,0,0,0,0,0,0,0,0)^T,\\
u_{13} &= 2842^{-1/2}\,(-9,-3,49,0,-9,-3,-9,0,0,-3,-9,0,-9,-3)^T.
\end{aligned}$$

The matrix representations of $\mathcal{A}_0|_U$, $\mathcal{A}_1|_U$ under this basis are $\mathcal{A}_0|_U = ((\mathcal{A}_0\,u_j)^T\,u_k)_{j,k=1}^{13}$ and $\mathcal{A}_1|_U = ((\mathcal{A}_1\,u_j)^T\,u_k)_{j,k=1}^{13}$, and a computation with Maple gives for the spectral norm

$$(\max\{\|\mathcal{A}_{\epsilon_1}|_U\,\mathcal{A}_{\epsilon_2}|_U\,\mathcal{A}_{\epsilon_3}|_U\|_2 : \epsilon_1,\epsilon_2,\epsilon_3 \in \{0,\,1\}\})^{1/3} < 0.95.$$

Hence Φ is continuous.

Let us prove the local linear independence of Φ. Here we use Theorem 2.1 and a procedure proposed by Goodman, Jia and Zhou [4]. The space $V \subset \mathbb{R}^{14}$ (as given in Section 2) is spanned by the vector $v_0 = (0,0,9/5,38/15,6/5,1,0,0,0,9/5,0,0,0,0)^T$ and by $\mathcal{A}_1 v_0$, $\mathcal{A}_0\mathcal{A}_1 v_0$, $\mathcal{A}_1\mathcal{A}_1 v_0$, $\mathcal{A}_0\mathcal{A}_0\mathcal{A}_1 v_0$, $\mathcal{A}_1\mathcal{A}_0\mathcal{A}_1 v_0$, $\mathcal{A}_0\mathcal{A}_0\mathcal{A}_0\mathcal{A}_1 v_0$, $\mathcal{A}_1\mathcal{A}_0\mathcal{A}_0\mathcal{A}_1 v_0$. Here v_0 is a right eigenvector of $\mathcal{A}_0$ to the eigenvalue 1. Hence $\dim V = 8$. Forming the matrix $\mathcal{M}$, we observe that the 7-th, the 9-th and the last four rows of $\mathcal{M}$ are zero rows. Hence $\operatorname{gsupp}\phi_1 = [0,3]$ and $\operatorname{gsupp}\phi_2 = [0,5]$. The remaining 8 rows of $\mathcal{M}$ are linearly independent. Thus Φ is linearly independent on $(0,1)$ by Theorem 2.1.

We can restrict our considerations to the shortened matrices $\widetilde{\mathcal{A}}_0$, $\widetilde{\mathcal{A}}_1$ as given in Example 5.1. Further, we can choose the matrix $\widetilde{\mathcal{M}}$ as the identity matrix. The procedure proposed in [4] gives $\operatorname{rank}\widetilde{\mathcal{A}}_0 = \operatorname{rank}\widetilde{\mathcal{A}}_0\widetilde{\mathcal{A}}_0 = \operatorname{rank}\widetilde{\mathcal{A}}_0\widetilde{\mathcal{A}}_1 = 7$ and the 7-th rows are zero; $\operatorname{rank}\widetilde{\mathcal{A}}_1 = \operatorname{rank}\widetilde{\mathcal{A}}_1\widetilde{\mathcal{A}}_1 = \operatorname{rank}\widetilde{\mathcal{A}}_1\widetilde{\mathcal{A}}_0 = 7$ and the 6-th rows are zero.

Hence, Φ is locally linearly independent. Moreover, ϕ_2 possesses a hole of length 1, namely $\phi_2(x) = 0$ for $x \in (5/2,\, 7/2)$.

7 Conclusions

In Section 3 we have presented an algorithm to compute the global supports of the r components of a compactly supported refinable function vector Φ from the refinement mask. The rest of the paper was restricted to $r = 2$.

While for the scalar case local linear independence of a refinable function ϕ guarantees that the support of ϕ is an integer interval without holes, this is not longer the case for $r > 1$. As we have seen in Section 4, a function vector $\Phi = (\phi_1,\, \phi_2)^T$ can only have holes if the lengths l_1 and l_2 of the global supports of ϕ_1, ϕ_2 satisfy $l_2/2 \le l_1 < l_2$. As another property, it has been shown that the endpoints of a hole cannot be integers. Further, Φ can have at most one biggest hole.

In Section 5 we have investigated matrices derived from the refinement mask. In Theorem 5.3 some results on the rank of these matrices are obtained leaving five different cases to be investigated. The first case has been solved completely in Theorem 5.4. The second case has been settled partially in Theorem 5.5. For the other cases we cannot give a final answer. However, if $\widetilde{\mathcal{A}}_0$ and $\widetilde{\mathcal{A}}_1$ have different rank (as in case (4) and case (5)) then one can show by Theorem 2.1 that Φ must have infinitely many holes. In case (4) this can be seen as follows. Since $\operatorname{rank}(\widetilde{\mathcal{A}}_0) = n-1$ it follows that $\operatorname{rank}(\widetilde{\mathcal{A}}_1^k\widetilde{\mathcal{A}}_0) = n-1$ for $k = 0,1,\ldots$. Hence, by Theorem 2.1, $\widetilde{\mathcal{A}}_1^k\widetilde{\mathcal{A}}_0$ has a zero row for all $k = 0,1,\ldots$ implying that Φ contains vanishing intervals of the form $(l_k + (2^k-1)/2^k,\, l_k + (2^k - 1/2)/2^k)$ with suitable integers l_k. Here l_k cannot be the same integer for all $k = 0,1,2,\ldots$, in particular one finds $l_k \ne l_{k+1}$, $k \in \mathbb{N}$. Hence Φ has infinitely many holes. This observation leads to the following

Conjecture 7.1 *Let $\Phi = (\phi_1, \phi_2)^T$ be a refinable, locally linearly independent vector of compactly supported, continuous functions. Then Φ cannot have more than one but finitely many holes.*

Our numerical computations however lead to the hypothesis that the cases (3), (4) and (5) contradict the property of local linear independence. So we obtain

Conjecture 7.2 *Let $\Phi = (\phi_1, \phi_2)^T$ be a refinable, locally linearly independent vector of compactly supported, continuous functions. Then Φ cannot have infinitely many holes.*

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Bibliography

1. C. de Boor, R. A. DeVore, and A. Ron, Approximation orders of FSI spaces in $L_2(\mathbb{R}^d)$, *Constr. Approx.* **14** (1998), 411–427.
2. H. L. Cheung, C. Tang, and D.-X. Zhou, Supports of locally linearly independent M-refinable functions, attractors of iterated function systems and tilings, preprint, 2001.
3. W. Dahmen and C. A. Micchelli, Biorthogonal wavelet expansions, *Constr. Approx.* **13** (1997), 293–328.
4. T. N. T. Goodman, R. Q. Jia, and D.-X. Zhou, Local linear independence of refinable function vectors of functions, *Proc. R. Soc. Edinb.* **130** (2000), 813–826.
5. T. A. Hogan, Stability and independence of the shifts of finitely many refinable functions, *J. Fourier Anal. Appl.* **3** (1997), 757-774.
6. K. Jetter and G. Plonka, A survey on L_2-Approximation order from shift-invariant spaces, in *Multivariate Approximation and Applications*, N. Dyn, D. Leviatan, D. Levin, and A. Pinkus (eds.), Cambridge University Press, 2001, 73–111.
7. R. Q. Jia, Shift-invariant spaces on the real line, *Proc. Amer. Math. Soc.* **125** (1997) 785–793.
8. R. Q. Jia and C. A. Micchelli, On linear independence of integer translates of a finite number of functions, *Proc. Edinburgh Math. Soc.* **36** (1992), 69–85.
9. R. Q. Jia, S. D. Riemenschneider and D.-X. Zhou, Vector subdivision schemes and multiple wavelets, *Math. Comp.* **67** (1998), 1533–1563.
10. G. Plonka and D.-X. Zhou, Properties of locally linearly independent refinable function vectors, preprint, 2001.
11. A. Ron and Z. Shen, The sobolev regularity of refinable functions, *J. Approx. Theory* **106** (2000), 185–225.
12. Q. Y. Sun, Two-scale difference equation: local and global linear independence, manuscript, 1991.
13. J.-Z. Wang, Linear independence relations of the shifts of a vector-valued distribution, manuscript, 2001.