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CONSTRUCTIVE APPROXIMATION

Generalized Spline Wavelets

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Abstract. A generalized multiresolution of multiplicity r, generated by r linearly independent spline functions with multiple knots, is introduced. With the help of the autocorrelation symbol and the two–scale symbol of the scaling functions the spline wavelets with multiple knots can completely be characterized. New decomposition and reconstruction algorithms, based on Fourier technique, are presented.

1. Introduction

Recently the concept of wavelets in $L^2(I\!R)$ was generalized in the following way. Let $\psi_0, \ldots, \psi_{r-1}$ ($r \in I\!N$) be functions in $L_2(I\!R)$ and let $\mathcal{B} := \{2^{j/2}\,\psi_\nu(2^j \cdot -l) : j, l \in Z\!\!Z, \nu = 0, \ldots, r-1\}$. Then $\psi_0, \ldots, \psi_{r-1}$ are called orthogonal wavelets of multiplicity r if $\mathcal{B}$ forms an orthonormal basis of $L^2(I\!R)$. We say that $\psi_0, \ldots, \psi_{r-1}$ are wavelets (prewavelets) of multiplicity r if $\mathcal{B}$ forms a Riesz basis of $L^2(I\!R)$ and $\psi_\nu(2^j \cdot -l)$ is orthogonal to $\psi_\mu(2^k \cdot -n)$ ($\nu, \mu \in \{0, \ldots, r-1\}$; $l, n, j, k \in Z\!\!Z$ with $j \neq k$).
The general theory of wavelets of multiplicity r is treated in [12, 13, 14]. As usual, the method is based on a generalization of the notion of multiresolution analysis as introduced by Mallat [20] and Meyer [21]. In [17], it is shown that any basis of orthogonal wavelets with multiplicity r composed with rapidly decaying wavelets is provided by such a generalized multiresolution of multiplicity r. For applications of multiwavelets for sparse representation of smooth linear operators we refer to [1]. In this paper the ideas will be used to construct spline wavelets with knots of multiplicity r.

In the following let $m \in I\!N_0$ and $1 \leq r \leq m+1$ be given integers. We consider equidistant knots of multiplicity r

$$x_l = x_l^r := \lfloor l/r \rfloor \quad (l \in Z\!\!Z), \tag{1.1}$$

where $\lfloor x \rfloor$ means the integer part of $x \in I\!R$.

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Let $N_k^{m,r} \in C^{m-r}(I\!R)$ $(1 \le r \le m+1;\ k \in Z\!\!\!Z)$ denote the normalized B-splines of degree m and defect r with the knots $x_k, \ldots, x_{k+m+1}$. Then we have

$$N_{\nu+rn}^{m,r} = N_\nu^{m,r}(\cdot - n) \quad (\nu = 0, \ldots, r-1;\ n \in Z\!\!\!Z).$$

We introduce the spline vector $\boldsymbol{N}_m^r := (N_\nu^{m,r})_{\nu=0}^{r-1}$. For $r = m+1$, the B-splines $N_\nu^{m,m+1}$ $(\nu = 0, \ldots, m)$ coincide with the well-known Bernstein polynomials. Using the ideas in [11, 12, 13] we shall consider the *generalized multiresolution* $\{V_j : j \in Z\!\!\!Z\}$ *of multiplicity* r of $L^2(I\!R)$ generated by the linearly independent scaling functions $N_\nu^{m,r}$ $(\nu = 0, \ldots, r-1)$, that is

$$(1.2)\quad V_j = V_j^{m,r} := \ \mathrm{clos}_{L^2}\left(\mathrm{span}\,\{N_\nu^{m,r}(2^j \cdot -l) :\ \nu = 0, \ldots, r-1;\ l \in Z\!\!\!Z\}\right).$$

In the following, for convenience the indices m, r are omitted relying on context. The sample space V_j $(j \in Z\!\!\!Z)$ is 2^{-j}–shift–invariant, i.e., for each $s \in V_j$ the translates $s(\cdot - 2^{-j}l)$ $(l \in Z\!\!\!Z)$ are also in V_j. Note that V_j can not be generated by less than r spline functions, it is not a principal shift–invariant space (cf. [6]). Let $\{W_j : j \in Z\!\!\!Z\}$ denote the sequence of wavelet spaces determined by the orthogonal complement of V_j in V_{j+1}

$$W_j = W_j^{m,r} := V_{j+1} \ominus V_j.$$

The 2^{-j}–shift–invariant wavelet space W_j $(j \in Z\!\!\!Z)$ can not be generated by less than r spline functions, too. Hence, the obtained wavelet decompositions are really new, and they do not depend on the known spline wavelet decompositions of multiplicity $r = 1$ (cf. [2, 7, 8, 9, 16]).
Different examples of spline wavelets of multiplicity r in the univariate case on the equidistant lattice can be found in [11, 12, 13, 22]. In [19] a more general case is considered, where the spline knots are nonuniform.

The purpose of this paper is a unified approach to univariate spline wavelets of multiplicity r on the equidistant lattice and to the corresponding decomposition and reconstruction algorithms based on Fourier technique. The basic tools of our method will be the two-scale symbols of scaling functions and wavelets.
The outline of the paper is as follows. In Section 2, we study the generalized multiresolution $\{V_j : j \in Z\!\!\!Z\}$ generated by the spline vector $\boldsymbol{N}_m^r$ in (1.2). In particular, we recall the two–scale relation and the autocorrelation symbol of $\boldsymbol{N}_m^r$. The Riesz basis property of $\mathcal{B}_j(\boldsymbol{N}_m^r) := \{2^{j/2}\, N_\nu^{m,r}(2^j \cdot -l) : \nu = 0, \ldots, r-1;\ l \in Z\!\!\!Z\}$ in V_j is equivalent to the assertion that the autocorrelation symbol of $\boldsymbol{N}_m^r$ is positive definite on the unit circle.
In Section 3, the wavelet spaces W_j $(j \in Z\!\!\!Z)$ are introduced. The wanted wavelet vectors $\boldsymbol{\psi}_m^r := (\psi_k^{m,r})_{k=0}^{r-1}$ are obtained by finding r independent functions in W_0 whose dilates and translates form a Riesz basis of W_j. Using the autocorrelation symbol of $\boldsymbol{N}_m^r$ the two–scale symbol $\boldsymbol{Q}_m^r$ of the wavelet vector $\boldsymbol{\psi}_m^r$ can completely be characterized.
Section 4 is devoted to new efficient decomposition and reconstruction algorithms based on periodization and Fourier technique. It turns out that the obtained algorithms are numerically stable if the Riesz stability of the bases in V_j and W_j

is assumed. Further, some efficient algorithms are presented which provide the needed input for the decomposition algorithm and handle the output of the decomposition and reconstruction algorithm, respectively.
In Sections 5 – 6, some special spline wavelets of multiplicity r are constructed. Spline wavelets obtained by $(m+1)$-th derivatives of cardinal Hermite fundamental splines are considered in Section 5 (cf. [22]). Finally, spline wavelets of multiplicity r with minimal support, firstly introduced by Goodman and Lee [12], are described in Section 6.

2. Multiresolution of multiplicity r

First we want to recall some basic properties of $\boldsymbol{N}_m^r$ which will be of use later. Therefore we need the following notions. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. The (r,r)-matrices $\boldsymbol{A}_k^r(z)$ ($k \in \mathbb{N}_0$, $r \in \mathbb{N}$, $z \in \mathbb{T}$) are defined for $k > r-1 \geq 0$ by

$$\boldsymbol{A}_k^r(z) := k \begin{pmatrix} \frac{1}{x_k} & -\frac{1}{x_{k+1}} & \dots & 0 & 0 \\ 0 & \frac{1}{x_{k+1}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{x_{k+r-2}} & -\frac{1}{x_{k+r-1}} \\ -\frac{z}{x_k} & 0 & \dots & 0 & \frac{1}{x_{k+r-1}} \end{pmatrix}, \tag{2.1}$$

where $x_{k+\nu}$ ($\nu = 0, \dots, r-1$) are given in (1.1). For $k = r-1 > 0$ let

$$\boldsymbol{A}_k^{k+1}(z) := k \begin{pmatrix} 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \\ -z & 0 & \dots & 0 & 1 \end{pmatrix} \tag{2.2}$$

and for $0 \leq k < r-1$

$$\boldsymbol{A}_k^r(z) := \begin{pmatrix} \boldsymbol{I}_{r-k-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_k^{k+1}(z) \end{pmatrix}, \tag{2.3}$$

where $\boldsymbol{A}_0^1(z) := 1 - z$. Here $\boldsymbol{I}_{r-k-1}$ denotes the $(r-k-1)$-th unit matrix and $\boldsymbol{0}$ a zero matrix. Note that $\det \boldsymbol{A}_k^r = c_k^r(1-z)$ with

$$c_k^r = \begin{cases} k^{k+1} & r \geq k+1, \\ \prod_{\nu=0}^{r-1} k/x_{k+\nu} & 1 \leq r \leq k. \end{cases} \tag{2.4}$$

In the following, let for an (r,r)-matrix $\boldsymbol{M}$

$$\sigma_1(\boldsymbol{M}) := \max\{|\lambda| : \boldsymbol{M}\boldsymbol{x} = \lambda\boldsymbol{x}\} \qquad (\boldsymbol{x} \neq \boldsymbol{0}),$$
$$\sigma_0(\boldsymbol{M}) := \min\{|\lambda| : \boldsymbol{M}\boldsymbol{x} = \lambda\boldsymbol{x}\} \qquad (\boldsymbol{x} \neq \boldsymbol{0}).$$

Note that $\sigma_1(\boldsymbol{M})$ is called the *spectral radius* of $\boldsymbol{M}$. Applying the Gershgorin circle theorem we find that all eigenvalues of $\boldsymbol{A}_k^r(z)$ are contained in the union $S := \cup_{\nu=0}^{r-1} S_\nu$ of the circles

$$S_\nu := \begin{cases} \{\alpha : \ |k/x_{k+\nu} - \alpha| \le k/x_{k+\nu}\} & 1 \le r \le k, \\ \{\alpha : \ |k - \alpha| \le k\} & r \ge k+1. \end{cases}$$

Hence, the eigenvalues of $\boldsymbol{A}_k^r(z)$ are bounded, and we have for $z \in \boldsymbol{T}$

$$\sigma_1(\boldsymbol{A}_k^r(z)) \le \begin{cases} 2kr/(k+1-r) & 1 \le r \le k, \\ 2k & r \ge k+1. \end{cases}$$

Now let $m \in I\!N_0$, $1 \le r \le m+1$. We put

$$\boldsymbol{D}_m^r := \boldsymbol{A}_m^r \, \boldsymbol{A}_{m-1}^r \cdots \boldsymbol{A}_0^r, \tag{2.5}$$

$$\boldsymbol{D}_{m,1}^r := \boldsymbol{A}_{2m+1}^r \, \boldsymbol{A}_{2m}^r \cdots \boldsymbol{A}_{m+1}^r. \tag{2.6}$$

Then it follows that $\det \boldsymbol{D}_m^r(z) = (1-z)^{m+1} \prod_{k=0}^m c_k^r$ and $\det \boldsymbol{D}_{m,1}^r(z) = (1-z)^{m+1} \prod_{k=m+1}^{2m+1} c_k^r$. The eigenvalues of $\boldsymbol{D}_m^r$ and $\boldsymbol{D}_{m,1}^r$ are bounded, and by (2.4) there are positive constants $c_{m,0}^r$, $d_{m,0}^r$ and $c_{m,1}^r$, $d_{m,1}^r$ such that for $z \in \boldsymbol{T}$ and $\mu = 0, 1$,

$$c_{m,0}^r \le \sigma_\mu \Big(\boldsymbol{D}_m^r(z) \, \boldsymbol{D}_m^r(z)^\star + \boldsymbol{D}_m^r(-z) \, \boldsymbol{D}_m^r(-z)^\star \Big) \le \ \mathrm{d}_{m,0}^r, \tag{2.7}$$

$$c_{m,1}^r \le \sigma_\mu \Big(\boldsymbol{D}_{m,1}^r(z) \, \boldsymbol{D}_{m,1}^r(z)^\star + \boldsymbol{D}_{m,1}^r(-z) \, \boldsymbol{D}_{m,1}^r(-z)^\star \Big) \le d_{m,1}^r. \tag{2.8}$$

The vector of Fourier transformed B-splines of degree m and defect r

$$\hat{N}_\nu^{m,r} := \int_{-\infty}^{\infty} N_\nu^{m,r}(x) \, e^{-i \cdot x} \, \mathrm{d}x \quad (\nu = 0, \ldots, r-1)$$

is denoted by $\hat{\boldsymbol{N}}_m^r := (\hat{N}_\nu^{m,r})_{\nu=0}^{r-1}$. In [23], it is shown that for $m \in I\!N_0$

$$(iu)^{m+1} \, \hat{\boldsymbol{N}}_m^r(u) = \boldsymbol{D}_m^r(e^{-iu}) \, \hat{\boldsymbol{N}}_{-1}^r(u) \quad (u \in I\!R) \tag{2.9}$$

with

$$\hat{\boldsymbol{N}}_{-1}^r(u) := \left(\frac{(iu)^{r-1}}{r-1}, \ldots, \frac{(iu)^1}{1}, 1 \right)^{\mathrm{T}} \tag{2.10}$$

and

$$(iu)^{m+1} \, \hat{\boldsymbol{N}}_{2m+1}^r(u) = \boldsymbol{D}_{m,1}^r(e^{-iu}) \, \hat{\boldsymbol{N}}_m^r(u) \quad (u \in I\!R). \tag{2.11}$$

The B-spline vector $\boldsymbol{N}_m^r$ satisfies a *two–scale relation* (or *refinement equation*)

$$\boldsymbol{N}_m^r = \sum_{l=-\infty}^{\infty} \boldsymbol{P}_{m,l}^r \, \boldsymbol{N}_m^r(2 \cdot -l), \tag{2.12}$$

where $(\boldsymbol{P}^r_{m,l})_{l=-\infty}^{\infty}$ are sequences of (r,r)-matrices with entries in l^1 (cf. [23]). The Fourier transformed two–scale relation of $\boldsymbol{N}^r_m$ reads

$$\hat{\boldsymbol{N}}^r_m = \boldsymbol{P}^r_m(e^{-i\cdot/2})\,\hat{\boldsymbol{N}}^r_m(\cdot/2) \tag{2.13}$$

with *the two–scale symbol* (or *refinement mask*) of $\boldsymbol{N}^r_m$

$$\boldsymbol{P}^r_m(z) := \frac{1}{2^{m+1}}\,\boldsymbol{D}^r_m(z^2)\,\boldsymbol{P}^r_{-1}\,\boldsymbol{D}^r_m(z)^{-1} \quad (z \in \boldsymbol{T}\setminus\{1\}) \tag{2.14}$$

and

$$\boldsymbol{P}^r_{-1} := \operatorname{diag}\,(2^{r-1},\ldots,2^0). \tag{2.15}$$

For $z=1$, the formula (2.14) is understood as limiting process

$$\boldsymbol{P}^r_m(1) := \frac{1}{2^{m+1}}\lim_{u\to 0}\boldsymbol{D}^r_m(e^{-2iu})\,\boldsymbol{P}^r_{-1}\,\boldsymbol{D}^r_m(e^{-iu})^{-1} \quad (u \in I\!R). \tag{2.16}$$

In particular, the two–scale symbol $\boldsymbol{P}^r_m(z)$ is a matrix polynomial of degree $\lfloor (m+1)/r\rfloor + 1$ in z with

$$\det\,\boldsymbol{P}^r_m(z) = 2^{-rm+r(r-3)/2}\,(1+z)^{m+1} \quad (z \in \boldsymbol{T}) \tag{2.17}$$

(cf. [23]).

The basis of our scaling functions $\mathcal{B}_j(\boldsymbol{N}^r_m) := \{2^{j/2}\,N^{m,r}_\nu(2^j\cdot - l) : \nu = 0,\ldots,r-1;\, l \in \mathbb{Z}\}$ satisfies the following Riesz basis property:

Lemma 2.1. *The basis $\mathcal{B}_j(\boldsymbol{N}^r_m)$ is a Riesz basis of V_j, i.e., there exist constants $0 < A \le B < \infty$ independent of j with*

$$A\sum_{l=-\infty}^{\infty}\|\boldsymbol{c}_l\|^2 \le \|\sum_{l=-\infty}^{\infty}\boldsymbol{c}_l^{\mathrm{T}}\,2^{j/2}\,\boldsymbol{N}^r_m(2^j\cdot - l)\|^2_{L^2} \le B\sum_{l=-\infty}^{\infty}\|\boldsymbol{c}_l\|^2 \tag{2.18}$$

for any sequence $\{\boldsymbol{c}_l\}_{l\in\mathbb{Z}}$ with $\boldsymbol{c}_l := (c_{l,\nu})_{\nu=0}^{r-1} \in \mathbb{C}^r$ and with the Euclidian norm $\|\boldsymbol{c}_l\|^2 := \sum_{\nu=0}^{r-1}|c_{l,\nu}|^2$. The best possible constants (Riesz constants) A, B in (2.18) *satisfy the following inequalities:*

$$\frac{(\lfloor m/r\rfloor + 1)}{2\,(m+1)^2\,9^m} \le A \le B \le \frac{\lfloor m/r\rfloor + 1}{m+1}.$$

For a proof of Lemma 2.1 with $j=0$, we refer to [3]. For $j \ne 0$, the assertion follows by scaling of $\boldsymbol{N}^r_m$.

Remark. The proof in [3] is made for more general knot sequences and for L^p. The lower bound for A given in Lemma 2.1 is not optimal. For more information on the condition of B–splines see also [4] and the references there. ♠

Consider the Hilbert space $L^2(I\!R)$ of all square integrable functions. For the function vectors $\boldsymbol{f} := (f_\nu)_{\nu=0}^{r-1}$, $\boldsymbol{g} := (g_\nu)_{\nu=0}^{r-1}$ $(f_\nu,\, g_\nu \in L^2(I\!R),\ \nu = 0, \ldots, r-1)$, we introduce the (r,r)-matrix

$$(2.19)\qquad (\boldsymbol{f}, \boldsymbol{g}) := (\langle f_\nu, g_\mu \rangle)_{\nu,\mu=0}^{r-1} = \left(\int_{-\infty}^{\infty} f_\nu(t)\, \overline{g_\mu(t)}\ \mathrm{d}t \right)_{\nu,\mu=0}^{r-1}.$$

Let the *autocorrelation symbol of* $\boldsymbol{N}_m^r$ be defined by

$$(2.20)\qquad \boldsymbol{\Phi}_m^r(z) := \sum_{l=-\infty}^{\infty} (\boldsymbol{N}_m^r(\cdot + l),\, \boldsymbol{N}_m^r)\, z^l \quad (z \in \boldsymbol{T}).$$

Applying the Poisson summation formula we have for $z = e^{-iu}$

$$(2.21)\qquad \boldsymbol{\Phi}_m^r(e^{-iu}) := \sum_{l=-\infty}^{\infty} \hat{\boldsymbol{N}}_m^r(u + 2\pi l) \hat{\boldsymbol{N}}_m^r(u + 2\pi l)^\star \quad (u \in I\!R)$$

with $(\hat{\boldsymbol{N}}_m^r)^\star := \overline{\hat{\boldsymbol{N}}_m^r}^{\mathrm{T}}$. Using (2.13), the relation

$$(2.22)\qquad \boldsymbol{\Phi}_m^r(z^2) = \boldsymbol{P}_m^r(z)\, \boldsymbol{\Phi}_m^r(z)\, \boldsymbol{P}_m^r(z)^\star + \boldsymbol{P}_m^r(-z)\, \boldsymbol{\Phi}_m^r(-z)\, \boldsymbol{P}_m^r(-z)^\star$$

is found. As in the case $r = 1$, the Riesz basis property of $\mathcal{B}_j(\boldsymbol{N}_m^r)$ is closely connected with the autocorrelation symbol of $\boldsymbol{N}_m^r$. The Riesz basis property (2.18) for $\mathcal{B}_j(\boldsymbol{N}_m^r)$ is equivalent to the assertion that the eigenvalues of $\boldsymbol{\Phi}_m^r(z)$ are bounded away from zero, i.e.,

$$(2.23)\qquad A \le \sigma_\mu(\boldsymbol{\Phi}_m^r(z)) \le B \quad (\mu = 0, 1;\ z \in \boldsymbol{T}).$$

In particular, the autocorrelation matrix $\boldsymbol{\Phi}_m^r(z)$ is positive definite for $z \in \boldsymbol{T}$ (cf. [13, 23]).

Remark. The bracket product in $I\!R^1$

$$[f,\, g] := \sum_{l=-\infty}^{\infty} f(\cdot + 2\pi l)\, \overline{g(\cdot + 2\pi l)} \quad (f, g \in L^2(I\!R)),$$

used for instance in [6, 15], can be generalized for our case in the following way. For $\boldsymbol{f} := (f_\nu)_{\nu=0}^{r-1}$, $\boldsymbol{g} := (g_\nu)_{\nu=0}^{r-1}$ $(f_\nu,\, g_\nu \in L^2(I\!R),\ \nu = 0, \ldots, r-1)$ let

$$[\boldsymbol{f},\, \boldsymbol{g}] := \sum_{l=-\infty}^{\infty} \boldsymbol{f}(\cdot + 2\pi l)\, \boldsymbol{g}(\cdot + 2\pi l)^\star.$$

Then we immediately obtain

$$\boldsymbol{\Phi}_m^r(e^{-i\cdot}) = [\hat{\boldsymbol{N}}_m^r,\, \hat{\boldsymbol{N}}_m^r]. \quad \spadesuit$$

For an analysis of the structure of $\boldsymbol{\Phi}_m^r$ the following *Euler–Frobenius matrix* $\boldsymbol{H}_{2m+1}^r$

$$(2.24)\qquad \boldsymbol{H}_{2m+1}^r(z) := \left(\sum_{l=-\infty}^{\infty} \mathrm{D}^\nu\, N_\mu^{2m+1,r}(x)|_{x=l}\, z^l \right)_{\nu,\mu=0}^{r-1} \qquad (z \in \boldsymbol{T})$$

is introduced, where D denotes the differentiation operator $\mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}\cdot}$. By Poisson summation formula we find

$$(2.25)\ \boldsymbol{H}_{2m+1}^r(e^{-iu}) = \sum_{l=-\infty}^{\infty} \left((i(u+2\pi l))^k\right)_{k=0}^{r-1} \hat{\boldsymbol{N}}_{2m+1}^r(u+2\pi l)^{\mathrm{T}} \quad (u \in I\!R).$$

In the case $r=1$, we obtain the well-known Euler–Frobenius polynomial

$$\boldsymbol{H}_{2m+1}^1(z) = \sum_{l=-\infty}^{\infty} N_{2m+1}(l)\, z^l \quad (z \in \boldsymbol{T}),$$

where $N_{2m+1} := \boldsymbol{N}_{2m+1}^1$ is the cardinal B–spline with simple knots. The Euler–Frobenius matrix plays a crucial role in solving cardinal and periodic Hermite spline interpolation problems (cf. [24]).
From (2.9), (2.11) and (2.21), it follows the relation

$$\begin{aligned}(2.26)\ \boldsymbol{\Phi}_m^r(z) &= \boldsymbol{D}_m^r(z)\, \boldsymbol{D}_r\, \overline{\boldsymbol{H}_{2m+1}^r(z)}\, (\boldsymbol{D}_{m,1}^r(z)^\star)^{-1} \quad (z \in \boldsymbol{T} \setminus \{1\}),\\ \boldsymbol{\Phi}_m^r(1) &= \lim_{u\to 0} \boldsymbol{D}_m^r(e^{-iu})\, \boldsymbol{D}_r\, \overline{\boldsymbol{H}_{2m+1}^r(e^{-iu})}\, (\boldsymbol{D}_{m,1}^r(e^{-iu})^\star)^{-1} \quad (u \in I\!R),\end{aligned}$$

where $\boldsymbol{D}_m^r$, $\boldsymbol{D}_{m,1}^r$ are defined in (2.5), (2.6) and

$$(2.27)\qquad \boldsymbol{D}_r := (-1)^{m+1} \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{(-1)^{r-1}}{r-1} \\ 0 & 0 & \dots & \frac{(-1)^{r-2}}{r-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & (-1)^1 & \dots & 0 & 0 \\ (-1)^0 & 0 & \dots & 0 & 0 \end{pmatrix} \qquad (r > 1),$$

$\boldsymbol{D}_1 := (-1)^{m+1}$ (cf. [23]). In particular, the autocorrelation matrix $\boldsymbol{\Phi}_m^r(z)$ is invertible on the unit circle $z \in \boldsymbol{T}$ if and only if the Euler–Frobenius matrix $\boldsymbol{H}_{2m+1}^r(z)$ is invertible on $z \in \boldsymbol{T}$. Now we can summarize:

Theorem 2.2. *Let $m \in I\!N_0$, $r \in I\!N$ with $1 \le r \le m+1$ be given. The sequence of spline spaces $\{V_j : j \in Z\!\!Z\}$ given by* (1.2) *forms a multiresolution analysis of multiplicity r, i.e., we have:*

(i) $V_j \subset V_{j+1} \quad (j \in Z\!\!Z)$.

(ii) $\mathrm{clos}_{L^2}\,(\bigcup_{j=-\infty}^{\infty} V_j) = L^2$.

(iii) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$.

(iv) $f \in V_j \Longleftrightarrow f(2\cdot) \in V_{j+1}$.

(v) *The basis $\mathcal{B}_j(\boldsymbol{N}_m^r)$ is a Riesz basis of V_j with* (2.18).

Proof. The relation (i) follows from the two–scale relation (2.12) of $\boldsymbol{N}_m^r$, and (iv) is an immediate consequence of the definition (1.2) of V_j. In order to verify (ii), we observe that $V_j = V_j^{m,r}$ contains $V_j^{m,1}$ ($j \in \mathbb{Z}$) which is generated by the cardinal B-spline $N_m := \boldsymbol{N}_m^1$ with simple knots. Thus, (ii) is satisfied, since

$$\operatorname{clos}_{L^2}\left(\bigcup_{j=-\infty}^{\infty} V_j^{m,1}\right) = L^2$$

(cf. [8, 9]). The Riesz basis property (v) is shown in Lemma 2.1. Finally, (v) leads to (iii) in the same manner as in [10], pp. 141 – 142. □

Example 2.1.
1. For $r = 1$, we obtain the known cardinal multiresolution analysis generated by N_m. We have with $x_k^1 = k$ and $z := e^{-iu}$,

$$\boldsymbol{A}_k^1(z) = 1 - z \quad (k \in I\!N_0), \quad \boldsymbol{D}_m^1(z) = \boldsymbol{D}_{m,1}^1(z) = (1-z)^{m+1},$$

$$(iu)^{m+1}\, \hat{N}_m(u) = (1 - e^{-iu})^{m+1}.$$

The two-scale symbol reads

$$\boldsymbol{P}_m^1(z) = \frac{(1-z^2)^{m+1}}{2^{m+1}\,(1-z)^{m+1}} = \left(\frac{1+z}{2}\right)^{m+1} \qquad (z \in \boldsymbol{T}).$$

For the autocorrelation symbol $\boldsymbol{\Phi}_m^1(z)$ we find

$$\boldsymbol{\Phi}_m^1(z) = \sum_{l=-\infty}^{\infty} N_{2m+1}(l+m+1)\, z^l,$$

such that

$$\boldsymbol{\Phi}_m^1(z) = \overline{\boldsymbol{\Phi}_m^1(z)} = \frac{(1-\overline{z})^{m+1}}{(1-z)^{m+1}}\,(-1)^{m+1}\boldsymbol{H}_{2m+1}^1(z) = z^{-m-1}\,\boldsymbol{H}_{2m+1}^1(z).$$

For $r = 1$, the Riesz constants A and B can be given explicitly (cf. [25]).
2. Let us consider the case $r = 2$. We obtain with $x_k^2 = \lfloor k/2 \rfloor$ for $k > 1$

$$\boldsymbol{A}_{2k}^2(z) = 2\begin{pmatrix} 1 & -1 \\ -z & 1 \end{pmatrix}, \quad \boldsymbol{A}_{2k+1}^2(z) = \begin{pmatrix} 2+1/k & -2+1/(k+1) \\ -z(2+1/k) & 2-1/(k+1) \end{pmatrix}.$$

In particular, for $m = 3$ and $z \in \boldsymbol{T}$, we have

$$\boldsymbol{D}_3^2(z) = 3\begin{pmatrix} 2+4z & -5+4z+z^2 \\ -4z-2z^2 & 1+4z-5z^2 \end{pmatrix}$$

and thus

$$\hat{\boldsymbol{N}}_3^2(u) = \frac{3}{(iu)^4}\begin{pmatrix} (2iu-5) + 4(iu+1)e^{-iu} + e^{-2iu} \\ 1 + 4(-iu+1)e^{-iu} + (-2iu-5)e^{-2iu} \end{pmatrix}.$$

The two–scale symbol reads

$$\boldsymbol{P}_3^2(z) = \frac{1}{16}\begin{pmatrix} 2+6z+z^2 & 5+2z \\ 2z+5z^2 & 1+6z+2z^2 \end{pmatrix}.$$

The autocorrelation symbol is given by

$$\boldsymbol{\Phi}_3^2(z) = \frac{1}{560}\begin{pmatrix} 9z^{-1}+128+9z & 53z^{-1}+80+z \\ z^{-1}+80+53z & 9z^{-1}+128+9z \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{\Phi}_3^2(e^{-iu})$

$$\lambda_0(\boldsymbol{\Phi}_3^2(e^{-iu})) = \frac{1}{560}(128+18\cos u - (9210+8640\cos u + 106\cos(2u))^{1/2})$$
$$\lambda_1(\boldsymbol{\Phi}_3^2(e^{-iu})) = \frac{1}{560}(128+18\cos u + (9210+8640\cos u + 106\cos(2u))^{1/2})$$

can be estimated for $z \in \boldsymbol{T}$ by

$$\frac{3}{140} = \lambda_0(\boldsymbol{\Phi}_3^2(1)) \le \lambda_0(\boldsymbol{\Phi}_3^2(z)) \le \lambda_0(\boldsymbol{\Phi}_3^2(-1)) = \frac{3}{20},$$
$$\frac{17}{70} = \lambda_1(\boldsymbol{\Phi}_3^2(-1)) \le \lambda_1(\boldsymbol{\Phi}_3^2(z)) \le \lambda_1(\boldsymbol{\Phi}_3^2(1)) = \frac{1}{2}.$$

Thus, we find the Riesz constants

$$A = \frac{3}{140}, \quad B = \frac{1}{2}.$$

With

$$\boldsymbol{H}_7^2(z) = \frac{1}{432}\begin{pmatrix} 37z+176z^2+3z^3 & 3z+176z^2+37z^3 \\ 175z-224z^2-21z^3 & 21z+224z^2-175z^3 \end{pmatrix}$$

and

$$\boldsymbol{D}_{3,1}^2(z) = \frac{35}{9}\begin{pmatrix} 6+26z+3z^2 & -17-18z \\ -18z-17z^2 & 3+26z+6z^2 \end{pmatrix},$$

the relation (2.26) can simply be verified. ♠

3. Wavelet spaces

Now we define the wavelet space W_j of level j ($j \in \mathbb{Z}$) as the orthogonal complement of V_j in V_{j+1}, i. e.

$$W_j = W_j^{m,r} := V_{j+1} \ominus V_j. \tag{3.1}$$

By $W_0 \subset V_1$, all elements of W_0 are cardinal spline functions of degree m and defect r on the lattice $\mathbb{Z}/2$.
Let $\psi_\nu^{m,r} \in W_0$ ($\nu = 0, \ldots, r-1$) and let $\boldsymbol{\psi}_m^r := (\psi_\nu^{m,r})_{\nu=0}^{r-1}$, $\hat{\boldsymbol{\psi}}_m^r := (\hat{\psi}_\nu^{m,r})_{\nu=0}^{r-1}$ be

the corresponding function vector and its Fourier transform. Then there exists a two–scale relation (or refinement equation) of the form

$$\boldsymbol{\psi}_m^r = \sum_{l=-\infty}^{\infty} \boldsymbol{Q}_{m,l}^r \, \boldsymbol{N}_m^r(2\cdot -l) \tag{3.2}$$

with a sequence $(\boldsymbol{Q}_{m,l}^r)_{l=-\infty}^{\infty}$ of (r,r)-matrices with entries in l^1. The Fourier transformed two–scale relation reads

$$\hat{\boldsymbol{\psi}}_m^r = \boldsymbol{Q}_m^r(e^{-i\cdot/2}) \, \hat{\boldsymbol{N}}_m^r(\cdot/2) \tag{3.3}$$

with the two–scale symbol (or refinement mask) of $\boldsymbol{\psi}_m^r$

$$\boldsymbol{Q}_m^r(z) := \frac{1}{2} \sum_{l=-\infty}^{\infty} \boldsymbol{Q}_{m,l}^r \, z^l \quad (z \in \mathbb{T}). \tag{3.4}$$

Further, we introduce the autocorrelation symbol $\boldsymbol{\Psi}_m^r$ of $\boldsymbol{\psi}_m^r$ by

$$\boldsymbol{\Psi}_m^r(z) := \sum_{l=-\infty}^{\infty} (\boldsymbol{\psi}_m^r(\cdot + l), \boldsymbol{\psi}_m^r) \, z^l, \tag{3.5}$$

where the (r,r)-matrix of scalar products $(\boldsymbol{\psi}_m^r(\cdot + l),\, \boldsymbol{\psi}_m^r)$ is defined as in (2.19). The following properties hold:

Lemma 3.1. *Let $m \in \mathbb{N}_0$, $1 \le r \le m+1$ be given. Then we have:*
(i) *For $u \in \mathbb{R}$,*

$$\boldsymbol{\Psi}_m^r(e^{-iu}) = [\hat{\boldsymbol{\psi}}_m^r,\, \hat{\boldsymbol{\psi}}_m^r](u) = \sum_{l=-\infty}^{\infty} \hat{\boldsymbol{\psi}}_m^r(u + 2\pi l) \, \hat{\boldsymbol{\psi}}_m^r(u + 2\pi l)^\star. \tag{3.6}$$

In particular, $\boldsymbol{\Psi}_m^r(z)$ $(z \in \mathbb{T})$ is Hermitian and positive semidefinite.
(ii) *For $z \in \mathbb{T}$,*

$$\boldsymbol{\Psi}_m^r(z^2) = \boldsymbol{Q}_m^r(z) \, \boldsymbol{\Phi}_m^r(z) \, \boldsymbol{Q}_m^r(z)^\star + \boldsymbol{Q}_m^r(-z) \, \boldsymbol{\Phi}_m^r(-z) \, \boldsymbol{Q}_m^r(-z)^\star. \tag{3.7}$$

Proof. The assertion (i) follows from the Poisson summation formula. By application of the two–scale relation (3.3) to (3.6) we find (3.7) in the usual manner (cf. [12]). $\square$

Now the following problem is of interest: How do we have to choose the two–scale symbol $\boldsymbol{Q}_m^r$ such that $\mathcal{B}_j(\boldsymbol{\psi}_m^r) := \{2^{j/2} \, \psi_\nu^{m,r}(2^j \cdot -l) : \nu = 0, \ldots, r-1;\ l \in \mathbb{Z}\}$ forms a Riesz basis of W_j?
We introduce the 2^{-j}-shift–invariant subspace of L^2 generated by $\boldsymbol{\psi}_m^r$

$$\begin{aligned} S_j(\boldsymbol{\psi}_m^r) &= \operatorname{clos}_{L^2} \operatorname{span} \mathcal{B}_j(\boldsymbol{\psi}_m^r) \\ &:= \operatorname{clos}_{L^2} \operatorname{span} \{\psi_\nu^{m,r}(2^j \cdot -l) : \nu = 0, \ldots, r-1,\ l \in \mathbb{Z}\}. \end{aligned} \tag{3.8}$$

By definition of $\boldsymbol{\psi}_m^r$ we have $S_0(\boldsymbol{\psi}_m^r) \subseteq W_0$. The function vector $\boldsymbol{\psi}_m^r$ is called a *wavelet vector (prewavelet vector)* if the following assumptions are satisfied:

(W1) $V_0 \perp S_0(\boldsymbol{\psi}_m^r)$.

(W2) There are constants $0 < C \le D < \infty$ such that

$$C \sum_{l=-\infty}^{\infty} \|\boldsymbol{d}_l\|^2 \le \| \sum_{l=-\infty}^{\infty} \boldsymbol{d}_l^{\mathrm{T}} \, \boldsymbol{\psi}_m^r(\cdot - l)\|_{L^2}^2 \le D \sum_{l=-\infty}^{\infty} \|\boldsymbol{d}_l\|^2$$

for any $\boldsymbol{d}_l := (d_{l,\nu})_{\nu=0}^{r-1} \in \mathbb{C}^r$.

(W3) $S_0(\boldsymbol{\psi}_m^r) = W_0$.

The assumptions (W1) – (W3) have the following consequences for the two–scale symbol $\boldsymbol{Q}_m^r$ of $\boldsymbol{\psi}_m^r$:

Theorem 3.2. *Let $m \in I\!N$, $1 \le r \le m+1$ be given. Then the two–scale symbol $\boldsymbol{Q}_m^r$ defines a wavelet vector $\boldsymbol{\psi}_m^r$ via* (3.3) *if and only if the following two conditions hold:*
(i) *For $z \in \boldsymbol{T}$,*

$$\boldsymbol{P}_m^r(z) \, \boldsymbol{\Phi}_m^r(z) \, \boldsymbol{Q}_m^r(z)^\star + \boldsymbol{P}_m^r(-z) \, \boldsymbol{\Phi}_m^r(-z) \, \boldsymbol{Q}_m^r(-z)^\star = \boldsymbol{0}. \tag{3.9}$$

(ii) *The eigenvalues of $(\boldsymbol{Q}_m^r(z) \, \boldsymbol{Q}_m^r(z)^\star + \boldsymbol{Q}_m^r(-z) \, \boldsymbol{Q}_m^r(-z)^\star)$ are bounded away from zero, i.e., there are constants $0 < \gamma \le \delta < \infty$ such that*

$$\gamma \le \sigma_\mu \Big(\boldsymbol{Q}_m^r(z) \, \boldsymbol{Q}_m^r(z)^\star + \boldsymbol{Q}_m^r(-z) \, \boldsymbol{Q}_m^r(-z)^\star \Big) \le \delta \quad (\mu = 0, 1; \; z \in \boldsymbol{T}) \tag{3.10}$$

and we have

$$\frac{C\,A}{B} \le A\,\gamma \le C \le D \le B\,\delta \le \frac{B\,D}{A} \tag{3.11}$$

with the Riesz constants A, B, C, D in (2.18) *and* (W2). *Furthermore,* (i) *and* (ii) *imply that the two–scale symbol matrix*

$$\boldsymbol{S}_m^r(z) := \begin{pmatrix} \boldsymbol{P}_m^r(z) & \boldsymbol{P}_m^r(-z) \\ \boldsymbol{Q}_m^r(z) & \boldsymbol{Q}_m^r(-z) \end{pmatrix} \quad (z \in \boldsymbol{T}) \tag{3.12}$$

is regular with

$$\frac{\min\{A,\, C\}}{B} \le \sigma_\mu \Big(\boldsymbol{S}_m^r(z) \, \boldsymbol{S}_m^r(z)^\star \Big) \le \frac{\max\{B,\, D\}}{A} \quad (\mu = 0, 1; \; z \in \boldsymbol{T}), \tag{3.13}$$

$$\boldsymbol{S}_m^r(z)^{-1} = \begin{pmatrix} \boldsymbol{\Phi}_m^r(z) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Phi}_m^r(-z) \end{pmatrix} \boldsymbol{S}_m^r(z)^\star \begin{pmatrix} \boldsymbol{\Phi}_m^r(z^2)^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Psi}_m^r(z^2)^{-1} \end{pmatrix}. \tag{3.14}$$

Proof. 1. The orthogonality relation (W1) is equivalent to

$$(\boldsymbol{N}_m^r(\cdot - l), \, \boldsymbol{\psi}_m^r) = \boldsymbol{0} \quad (l \in \mathbb{Z}).$$

By Poisson summation formula and using the relations (2.13), (3.3) and (2.21) this leads to

$$\begin{aligned}
\mathbf{0} &= \sum_{l=-\infty}^{\infty} \hat{\boldsymbol{N}}_m^r(u+2\pi l)\, \hat{\boldsymbol{\psi}}_m^r(u+2\pi l)^\star \\
&= \sum_{l=-\infty}^{\infty} \boldsymbol{P}_m^r(e^{-i(u/2+\pi l)})\, \hat{\boldsymbol{N}}_m^r(u/2+\pi l)\, \hat{\boldsymbol{N}}_m^r(u/2+\pi l)^\star\, \boldsymbol{Q}_m^r(e^{-i(u/2+\pi l)})^\star \\
&= \boldsymbol{P}_m^r(e^{-iu/2})\, \boldsymbol{\Phi}_m^r(e^{-iu/2})\, \boldsymbol{Q}_m^r(e^{-iu/2})^\star \\
&\quad + \boldsymbol{P}_m^r(-e^{-iu/2})\, \boldsymbol{\Phi}_m^r(-e^{-iu/2})\, \boldsymbol{Q}_m^r(-e^{-iu/2})^\star,
\end{aligned}$$

i.e., (W1) is equivalent to (3.9).
2. As in the sample space, the Riesz basis property (W2) with Riesz bounds C and D is equivalent to the assertion that the eigenvalues of the autocorrelation symbol $\boldsymbol{\Psi}_m^r(z)$ are bounded away from zero, i.e., for $z \in \boldsymbol{T}$ we have

$$C \le \sigma_\mu(\boldsymbol{\Psi}_m^r(z)) \le D \qquad (\mu = 0,1). \tag{3.15}$$

Since $\boldsymbol{\Phi}_m^r(z)$ is positive definite, both terms on the right hand side of (3.7) are positive semidefinite. Thus from (3.15), it follows by (2.23) that for all $z \in \boldsymbol{T}$

$$\frac{C}{B} \le \sigma_\mu\Big(\boldsymbol{Q}_m^r(z)\, \boldsymbol{Q}_m^r(z)^\star + \boldsymbol{Q}_m^r(-z)\, \boldsymbol{Q}_m^r(-z)^\star\Big) \le \frac{D}{A} \quad (\mu = 0,1).$$

Vice versa, (3.10) implies that by (2.23) for $z \in \boldsymbol{T}$ and $\mu = 0,1$

$$A\,\gamma \le \sigma_\mu(\boldsymbol{\Psi}_m^r(z)) \le B\,\delta.$$

Hence, we have (3.11).
3. Using the relations (2.22), (3.7) and (3.9) we obtain

$$\boldsymbol{S}_m^r(z) \begin{pmatrix} \boldsymbol{\Phi}_m^r(z) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}_m^r(-z) \end{pmatrix} \boldsymbol{S}_m^r(z)^\star = \begin{pmatrix} \boldsymbol{\Phi}_m^r(z^2) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}_m^r(z^2) \end{pmatrix}. \tag{3.16}$$

Thus, we find by (2.23) and (3.15) for $z \in \boldsymbol{T}$

$$\frac{\min\{A,\,C\}}{B} \le \sigma_\mu\Big(\boldsymbol{S}_m^r(z)\, \boldsymbol{S}_m^r(z)^\star\Big) \le \frac{\max\{B,\,D\}}{A} \quad (\mu = 0,1).$$

The relation (3.14) is a simple consequence of (3.16).
4. By (3.9) it is ensured that $S_0(\boldsymbol{\psi}_m^r) \subseteq W_0$. Take $\psi_0 \in W_0$ with $\psi_0 \perp S_0(\boldsymbol{\psi}_m^r)$. Then by $W_0 \subset V_1$, there exists a vector $\boldsymbol{C} := (C_\nu)_{\nu=0}^{r-1}$, $C_\nu \in L_{2\pi}^2\, (\nu = 0, \ldots r-1)$ with

$$\hat{\psi}_0 = \boldsymbol{C}(e^{-i\cdot/2})^{\mathrm{T}}\, \hat{\boldsymbol{N}}_m^r(\cdot/2).$$

By $\psi_0 \perp S_0(\boldsymbol{\psi}_m^r)$ and $\psi_0 \perp V_0$ we find for $z \in \boldsymbol{T}$ in the same manner as before

$$\begin{aligned}
\boldsymbol{P}_m^r(z)\boldsymbol{\Phi}_m^r(z)\overline{\boldsymbol{C}(z)} + \boldsymbol{P}_m^r(-z)\, \boldsymbol{\Phi}_m^r(-z)\, \overline{\boldsymbol{C}(-z)} = \mathbf{0}, \\
\boldsymbol{Q}_m^r(z)\boldsymbol{\Phi}_m^r(z)\overline{\boldsymbol{C}(z)} + \boldsymbol{Q}_m^r(-z)\, \boldsymbol{\Phi}_m^r(-z)\, \overline{\boldsymbol{C}(-z)} = \mathbf{0},
\end{aligned}$$

i.e.,

$$\boldsymbol{S}_m^r(z)\begin{pmatrix}\boldsymbol{\Phi}_m^r(z)\,\overline{\boldsymbol{C}(z)}\\ \boldsymbol{\Phi}_m^r(-z)\,\overline{\boldsymbol{C}(-z)}\end{pmatrix}=\mathbf{0}.$$

Hence, by (3.13) and (2.23) we have $\boldsymbol{C}(z)=\mathbf{0}$, that is $\psi_0=0$. Thus, $S_0(\boldsymbol{\psi}_m^r)=W_0$. □

Remark. We show that W_0 can not be spanned by a wavelet vector of less than r different spline functions: Assume that r spline functions $\psi_\nu^{m,r}\in W_0$ ($\nu=0,\ldots,r-1$) generating a Riesz basis of W_0 can be formed by linear independent integer translates of $r-n$ ($1\le n\le r-1$) spline functions $\tilde\psi_\mu\in W_0$ ($\mu=0,\ldots,r-n-1$). The linear independence of $\psi_\nu^{m,r}(\cdot-l)$ ($\nu=0,\ldots,r-1$; $l\in\mathbb{Z}$) yields, that the functions $\psi_\nu^{m,r}$ can not represented by *integer* translates of $\tilde\psi_\mu$ ($\mu=0,\ldots,r-n-1$) only. Since the lattice $\mathbb{Z}/2$ is fixed, it follows that there is at least one function $\psi:=\tilde\psi_\mu$ ($\mu\in\{0,\ldots,r-n-1\}$) whose half integer translates $\psi(\cdot-l/2)$ ($l\in\mathbb{Z}$) are contained in W_0. By $\psi\in V_1$, there is a vector $\boldsymbol{C}=(C_\nu)_{\nu=0}^{r-1}$ ($C_\nu\in L^2_{2\pi}$, $\nu=0,\ldots,r-1$) with

$$\hat\psi=\boldsymbol{C}(e^{-i\cdot/2})^{\mathrm{T}}\,\hat{\boldsymbol{N}}_m^r(\cdot/2). \tag{3.17}$$

From

$$\langle\psi(\cdot+l/2),N_k^{m,r}\rangle=0\quad(k=0,\ldots,r-1,\ l\in\mathbb{Z})$$

it follows by Poisson summation formula that

$$\sum_{n=-\infty}^{\infty}(-1)^{nl}\overline{\hat\psi(u+2\pi n)}\,\hat{\boldsymbol{N}}_m^r(u+2\pi n)=\mathbf{0}\quad(l\in\mathbb{Z}).$$

Hence, using the two–scale relations (2.13) and (3.17),

$$\boldsymbol{P}_m^r(z)\,\boldsymbol{\Phi}_m^r(z)\,\overline{\boldsymbol{C}(z)}=\mathbf{0}\quad(z\in\boldsymbol{T}),$$

such that we have $\boldsymbol{C}(z)=\mathbf{0}$ ($z\in\boldsymbol{T}\setminus\{1\}$). But this is a contradiction to the assumption that $\psi(\cdot-l/2)$ ($l\in\mathbb{Z}$) are linearly independent. ♠

With the help of the conditions (i) and (ii) of Theorem 3.2 the two–scale symbol $\boldsymbol{Q}_m^r$ of the wavelet vector $\boldsymbol{\psi}_m^r$ can be described more exactly.

Theorem 3.3. *Let $m\in\mathbb{N}_0$, $1\le r\le m+1$ be fixed. The (r,r)-matrix $\boldsymbol{Q}_m^r:\boldsymbol{T}\to\mathbb{C}^{r\times r}$ is a two–scale symbol of a wavelet vector $\boldsymbol{\psi}_m^r$ if and only if $\boldsymbol{Q}_m^r$ is of the form*

$$\boldsymbol{Q}_m^r(z)=z\boldsymbol{K}(z^2)\,\boldsymbol{D}_m^r(z)^\star\,\boldsymbol{\Phi}_m^r(z)^{-1}\quad(z\in\boldsymbol{T}) \tag{3.18}$$

with a matrix $\boldsymbol{K}:\boldsymbol{T}\to\mathbb{C}^{r\times r}$ whose elements lie in the Wiener class and whose eigenvalues satisfy for some positive α, β the relation

$$0<\alpha\le\sigma_\mu(\boldsymbol{K}(z))\le\beta<\infty\quad(\mu=0,1;\ z\in\boldsymbol{T}). \tag{3.19}$$

Proof. 1. Assume that $\boldsymbol{Q}_m^r$ is given in the form (3.18) with (3.19). Then, by definition of $\boldsymbol{D}_m^r$ and $\boldsymbol{\Phi}_m^r$, the elements of $\boldsymbol{Q}_m^r$ lie in the Wiener class. By (2.14) and (3.18) we have

$$\boldsymbol{P}_m^r(z)\,\boldsymbol{\Phi}_m^r(z)\,\boldsymbol{Q}_m^r(z)^\star = 2^{-m-1}\,z^{-1}\,\boldsymbol{D}_m^r(z^2)\,\boldsymbol{P}_{-1}^r\,\boldsymbol{K}(z^2)^\star \quad (z \in \mathbb{T}).$$

Thus, (3.9) is satisfied. The assumption (3.10) immediately follows by (3.18)–(3.19), (2.23) and (2.7).

2. Let $\boldsymbol{Q}_m^r : \mathbb{T} \to \mathbb{C}^{r\times r}$ be an (r,r)-matrix with elements in the Wiener class and satisfying the conditions (3.9) – (3.10). The relation (3.9) yields by (2.14) for $z \in \mathbb{T} \setminus \{-1,1\}$

$$\boldsymbol{D}_m^r(z)^{-1}\,\boldsymbol{\Phi}_m^r(z)\,\boldsymbol{Q}_m^r(z)^\star + \boldsymbol{D}_m^r(-z)^{-1}\,\boldsymbol{\Phi}_m^r(-z)\,\boldsymbol{Q}_m^r(-z)^\star = \boldsymbol{0}, \tag{3.20}$$

i.e.,

$$\boldsymbol{\Phi}_m^r(-z)\,\boldsymbol{Q}_m^r(-z)^\star = -\boldsymbol{D}_m^r(-z)\,\boldsymbol{D}_m^r(z)^{-1}\,\boldsymbol{\Phi}_m^r(z)\,\boldsymbol{Q}_m^r(z)^\star$$

and

$$\boldsymbol{Q}_m^r(-z) = -\boldsymbol{Q}_m^r(z)\,\boldsymbol{\Phi}_m^r(z)\,(\boldsymbol{D}_m^r(z)^\star)^{-1}\,\boldsymbol{D}_m^r(-z)^\star\,\boldsymbol{\Phi}_m^r(-z)^{-1}.$$

Hence, by (3.7) the autocorrelation symbol of $\boldsymbol{\psi}_m^r$ reads for $z \in \mathbb{T} \setminus \{1,-1\}$

$$\begin{aligned}\boldsymbol{\Psi}_m^r(z^2) &= \Big(\boldsymbol{Q}_m^r(z)\,\boldsymbol{D}_m^r(z) - \boldsymbol{Q}_m^r(-z)\,\boldsymbol{D}_m^r(-z)\Big)\,\boldsymbol{D}_m^r(z)^{-1}\,\boldsymbol{\Phi}_m^r(z)\,\boldsymbol{Q}_m^r(z)^\star \\ &= \boldsymbol{Q}_m^r(z)\,\boldsymbol{\Phi}_m^r(z)\,(\boldsymbol{D}_m^r(z)^\star)^{-1}\Big[\boldsymbol{D}_m^r(z)^\star\,\boldsymbol{\Phi}_m^r(z)^{-1}\,\boldsymbol{D}_m^r(z) \\ &\quad + \boldsymbol{D}_m^r(-z)^\star\,\boldsymbol{\Phi}_m^r(-z)^{-1}\,\boldsymbol{D}_m^r(-z)\Big]\,\boldsymbol{D}_m^r(z)^{-1}\,\boldsymbol{\Phi}_m^r(z)\,\boldsymbol{Q}_m^r(z)^\star.\end{aligned} \tag{3.21}$$

In particular, we have

$$\begin{aligned}\boldsymbol{Q}_m^r(z)\,\boldsymbol{D}_m^r(z) - \boldsymbol{Q}_m^r(-z)\,\boldsymbol{D}_m^r(-z) &= \boldsymbol{Q}_m^r(z)\,\boldsymbol{\Phi}_m^r(z)\,(\boldsymbol{D}_m^r(z)^\star)^{-1} \\ &\times\Big[\boldsymbol{D}_m^r(z)^\star\,\boldsymbol{\Phi}_m^r(z)^{-1}\,\boldsymbol{D}_m^r(z) + \boldsymbol{D}_m^r(-z)^\star\,\boldsymbol{\Phi}_m^r(-z)^{-1}\,\boldsymbol{D}_m^r(-z)\Big].\end{aligned} \tag{3.22}$$

By (2.23) and (2.7), the eigenvalues of

$$\boldsymbol{D}_m^r(z)^\star\,\boldsymbol{\Phi}_m^r(z)^{-1}\,\boldsymbol{D}_m^r(z) + \boldsymbol{D}_m^r(-z)^\star\,\boldsymbol{\Phi}_m^r(-z)^{-1}\,\boldsymbol{D}_m^r(-z)$$

are bounded away from zero for $z \in \mathbb{T}$. Thus, from (3.15) and (3.21) it follows that the eigenvalues of $\boldsymbol{Q}_m^r(z)\,\boldsymbol{\Phi}_m^r(z)\,(\boldsymbol{D}_m^r(z)^\star)^{-1}$ are bounded away from zero for $z \in \mathbb{T}$, too. We put for $z \in \mathbb{T}$

$$z^{-1}\,\boldsymbol{K}(z^2)^\star := \Big(\boldsymbol{Q}_m^r(z)\,\boldsymbol{D}_m^r(z) - \boldsymbol{Q}_m^r(-z)\,\boldsymbol{D}_m^r(-z)\Big)^{-1}\,\boldsymbol{\Psi}_m^r(z^2). \tag{3.23}$$

Then by (3.22) it follows that

$$\begin{aligned}z^{-1}\,\boldsymbol{K}(z^2)^\star = \Big(&\boldsymbol{Q}_m^r(z)\,\boldsymbol{\Phi}_m^r(z)\,(\boldsymbol{D}_m^r(z)^\star)^{-1}\,[\boldsymbol{D}_m^r(z)^\star\,\boldsymbol{\Phi}_m^r(z)^{-1}\,\boldsymbol{D}_m^r(z) \\ &+ \boldsymbol{D}_m^r(-z)^\star\,\boldsymbol{\Phi}_m^r(-z)^{-1}\,\boldsymbol{D}_m^r(-z)]\Big)^{-1}\,\boldsymbol{\Psi}_m^r(z^2).\end{aligned}$$

From the considerations above it follows that there are constants α, β with

$$0 < \alpha \le \sigma_\mu(\boldsymbol{K}(z)) \le \beta < \infty \quad (\mu = 0, 1).$$

By (3.21) we find for $z \in \mathbb{T}$

$$\boldsymbol{Q}_m^r(z)^\star = z^{-1}\,\boldsymbol{\Phi}_m^r(z)^{-1}\,\boldsymbol{D}_m^r(z)\,\boldsymbol{K}(z^2)^\star,$$

i.e.,

$$\boldsymbol{Q}_m^r(z) = z\,\boldsymbol{K}(z^2)\,\boldsymbol{D}_m^r(z)^\star\,\boldsymbol{\Phi}_m^r(z)^{-1},$$

where for $z = 1, -1$, the formula for $\boldsymbol{Q}_m^r(z)$ follows by limiting process. □

With the help of (2.26) we find by $\boldsymbol{\Phi}_m^r(z) = \boldsymbol{\Phi}_m^r(z)^\star$

$$\boldsymbol{\Phi}_m^r(z)^{-1} = (\boldsymbol{D}_m^r(z)^\star)^{-1}\,(\boldsymbol{D}_r^{\mathrm{T}})^{-1}(\boldsymbol{H}_{2m+1}^r(z)^{\mathrm{T}})^{-1}\,\boldsymbol{D}_{m,1}^r(z),$$

such that the two–scale symbol $\boldsymbol{Q}_m^r$ in Theorem 3.3 can also be written as

$$\boldsymbol{Q}_m^r(z) = z\,\tilde{\boldsymbol{K}}(z^2)\,(\boldsymbol{H}_{2m+1}^r(z)^{\mathrm{T}})^{-1}\,\boldsymbol{D}_{m,1}^r(z)$$

with

$$\tilde{\boldsymbol{K}}(z^2) := \boldsymbol{K}(z^2)\,(\boldsymbol{D}_r^{\mathrm{T}})^{-1}.$$

Thus, the conditions (3.18) – (3.19) are equivalent to

$$\boldsymbol{Q}_m^r(z) = z\,\tilde{\boldsymbol{K}}(z^2)\,(\boldsymbol{H}_{2m+1}^r(z)^{\mathrm{T}})^{-1}\,\boldsymbol{D}_{m,1}^r(z) \tag{3.24}$$

with a matrix $\tilde{\boldsymbol{K}} : \mathbb{T} \to \mathbb{C}^{r\times r}$, whose elements lie in the Wiener class and with

$$0 < \tilde{\alpha} \le \sigma_\mu(\tilde{\boldsymbol{K}}(z)) \le \tilde{\beta} < \infty \quad (\mu = 0, 1;\ z \in \mathbb{T}). \tag{3.25}$$

The computation of $\boldsymbol{H}_{2m+1}^r$ is simpler than getting $\boldsymbol{\Phi}_m^r$ and thus, (3.24) – (3.25) will be prefered. In fact, (3.24) with (2.11) shows that all wavelets can be obtained by taking derivatives of splines of double order. In Sections 5 – 6 we shall construct some special wavelets by appropriate choices of the matrix $\tilde{\boldsymbol{K}}$.

4. Decomposition and reconstruction algorithms

In this section we derive efficient decomposition and reconstruction algorithms based on periodization and fast Fourier transform.
In order to decompose a given function $f_{j+1} \in V_{j+1} \bigcap L^1(\mathbb{R})$ $(j \in \mathbb{Z})$ of the form

$$f_{j+1} = \sum_{l=-\infty}^{\infty} \boldsymbol{c}_{j+1,l}^{\mathrm{T}}\,\boldsymbol{N}_m^r(2^{j+1}\cdot - l) \tag{4.1}$$

with $\boldsymbol{c}_{j+1,l} := (c^{\nu}_{j+1,l})_{\nu=0}^{r-1}$ and $(c^{\nu}_{j+1,l})_{l=-\infty}^{\infty} \in l^1$ $(\nu = 0,\ldots,r-1)$, we have to find the uniquely determined functions $f_j \in V_j$ and $g_j \in W_j$ such that

$$f_{j+1} = f_j + g_j. \tag{4.2}$$

The wanted functions $f_j \in V_j$ and $g_j \in W_j$ can be uniquely represented by

$$f_j = \sum_{l=-\infty}^{\infty} \boldsymbol{c}_{j,l}^{\mathrm{T}} \, \boldsymbol{N}_m^r(2^j \cdot -l), \quad g_j = \sum_{l=-\infty}^{\infty} \boldsymbol{d}_{j,l}^{\mathrm{T}} \, \boldsymbol{\psi}_m^r(2^j \cdot -l), \tag{4.3}$$

where the coefficient vectors $\boldsymbol{c}_{j,l} := (c^{\nu}_{j,l})_{\nu=0}^{r-1}$, $\boldsymbol{d}_{j,l} := (d^{\nu}_{j,l})_{\nu=0}^{r-1}$ with $(c^{\nu}_{j,l})_{l=-\infty}^{\infty}$, $(d^{\nu}_{j,l})_{l=-\infty}^{\infty} \in l^1$ $(\nu = 0,\ldots,r-1)$ are unknown.
In order to reconstruct $f_{j+1} \in V_{j+1}$ $(j \in I\!N_0)$, we have to compute the sum (4.2) with given $f_j \in V_j$ and $g_j \in W_j$.
Let $\hat{f}_j$, $\hat{g}_j$, $\hat{f}_{j+1}$ be the Fourier transforms of f_j, g_j, f_{j+1}, and let for $z \in \boldsymbol{T}$

$$\begin{aligned} \boldsymbol{C}_j(z) &:= \sum_{l=-\infty}^{\infty} \boldsymbol{c}_{j,l}\, z^l, \quad \boldsymbol{D}_j(z) := \sum_{l=-\infty}^{\infty} \boldsymbol{d}_{j,l}\, z^l, \\ \boldsymbol{C}_{j+1}(z) &:= \sum_{l=-\infty}^{\infty} \boldsymbol{c}_{j+1,l}\, z^l. \end{aligned} \tag{4.4}$$

Hence, for $u \in I\!R$ it follows

$$\begin{aligned} \hat{f}_j(u) &= 2^{-j}\, \boldsymbol{C}_j(e^{-iu/2^j})^{\mathrm{T}}\, \hat{\boldsymbol{N}}_m^r(2^{-j}u), \\ \hat{g}_j(u) &= 2^{-j}\, \boldsymbol{D}_j(e^{-iu/2^j})^{\mathrm{T}}\, \hat{\boldsymbol{\psi}}_m^r(2^{-j}u), \\ \hat{f}_{j+1}(u) &= 2^{-j-1}\, \boldsymbol{C}_{j+1}(e^{-iu/2^{j+1}})^{\mathrm{T}}\, \hat{\boldsymbol{N}}_m^r(2^{-j-1}u). \end{aligned} \tag{4.5}$$

We obtain the following

Theorem 4.1. *Let $f_{j+1} \in V_{j+1}$, $f_j \in V_j$, $g_j \in W_j$ with* (4.1) *and* (4.3) *be given. Then*

$$\begin{pmatrix} \boldsymbol{C}_{j+1}(z) \\ \boldsymbol{C}_{j+1}(-z) \end{pmatrix}^{\mathrm{T}} = 2 \begin{pmatrix} \boldsymbol{C}_j(z^2) \\ \boldsymbol{D}_j(z^2) \end{pmatrix}^{\mathrm{T}} \boldsymbol{S}_m^r(z) \quad (z \in \boldsymbol{T}) \tag{4.6}$$

implies that $f_{j+1} = f_j + g_j$. Moreover, for known $\boldsymbol{C}_j(z)$, $\boldsymbol{D}_j(z)$, the function vector $\boldsymbol{C}_{j+1}(z)$ $(z \in \boldsymbol{T})$ is uniquely determined by (4.6). *Vice versa, for known $\boldsymbol{C}_{j+1}(z)$, the function vectors $\boldsymbol{C}_j(z)$, $\boldsymbol{D}_j(z)$ $(z \in \boldsymbol{T})$ can uniquely be computed be* (4.6).

Proof. The relation (4.6) implies that

$$\boldsymbol{C}_{j+1}(z)^{\mathrm{T}} = 2(\boldsymbol{C}_j(z^2)^{\mathrm{T}}\, \boldsymbol{P}_m^r(z) + \boldsymbol{D}_j(z^2)^{\mathrm{T}}\, \boldsymbol{Q}_m^r(z)). \tag{4.7}$$

Putting $z := e^{-iu/2^{j+1}}$ and multiplying (4.7) with $2^{-j-1}\, \hat{\boldsymbol{N}}_m^r(2^{-j-1}u)$, we obtain

$$\begin{aligned} 2^{-j-1}\boldsymbol{C}_{j+1}(e^{-iu/2^{j+1}})^{\mathrm{T}} \;\; & \hat{\boldsymbol{N}}_m^r(2^{-j-1}u) \\ &= 2^{-j}\, \boldsymbol{C}_j(e^{-iu/2^j})^{\mathrm{T}}\, \boldsymbol{P}_m^r(e^{-iu/2^{j+1}})\, \hat{\boldsymbol{N}}_m^r(2^{-j-1}u) \\ &+ 2^{-j}\, \boldsymbol{D}_j(e^{-iu/2^j})^{\mathrm{T}}\, \boldsymbol{Q}_m^r(e^{-iu/2^{j+1}})\, \hat{\boldsymbol{N}}_m^r(2^{-j-1}u). \end{aligned}$$

Hence, applying the two–scale relations (2.13), (3.3) we have by (4.5)

$$f_{j+1} = f_j + g_j.$$

For known $\boldsymbol{C}_j(z)$, $\boldsymbol{D}_j(z)$, the vector $\boldsymbol{C}_{j+1}(z)$ $(z \in \mathbb{T})$ can be computed by (4.7). Since the two–scale symbol matrix $\boldsymbol{S}_m^r(z)$ is invertible for $z \in \mathbb{T}$, we find from (4.6)

$$\begin{pmatrix} \boldsymbol{C}_j(z^2) \\ \boldsymbol{D}_j(z^2) \end{pmatrix}^{\mathrm{T}} = \frac{1}{2} \begin{pmatrix} \boldsymbol{C}_{j+1}(z) \\ \boldsymbol{C}_{j+1}(-z) \end{pmatrix}^{\mathrm{T}} \boldsymbol{S}_m^r(z)^{-1},$$

such that $\boldsymbol{C}_j(z^2)$, $\boldsymbol{D}_j(z^2)$ can simply be computed for known $\boldsymbol{C}_{j+1}(z)$ $(z \in \mathbb{T})$. $\square$

In order to derive efficient decomposition and reconstruction algorithms we want to periodize the functions f_{j+1}, f_j, g_j in the following way.
Let $N \in \mathbb{N}$, $N_j := 2^j N$. In practice, we can suppose that these functions are approximately zero on $\mathbb{R} \setminus [-N/2, N/2]$ for some $N \in \mathbb{N}$. Hence, the N-periodization of $f_{j+1} \in V_{j+1}$

$$\tilde{f}_{j+1} := \sum_{n=-\infty}^{\infty} f_{j+1}(\cdot + nN)$$

is a good approximation of f_{j+1} in $[-N/2, N/2]$. With

$$\tilde{\boldsymbol{c}}_{j+1,l} := \sum_{n=-\infty}^{\infty} \boldsymbol{c}_{j+1,l+nN_{j+1}} \quad (l = 0, \ldots, N_{j+1} - 1),$$

$$\tilde{\boldsymbol{N}}_{m,j+1}^r := \sum_{n=-\infty}^{\infty} \boldsymbol{N}_m^r(2^{j+1} \cdot + nN_{j+1}),$$

we obtain by (4.1)

$$\tilde{f}_{j+1} = \sum_{l=0}^{N_{j+1}-1} \tilde{\boldsymbol{c}}_{j+1,l}^{\mathrm{T}} \, \tilde{\boldsymbol{N}}_{m,j+1}^r(\cdot - 2^{-j-1} l).$$

The functions $f_j \in V_j$ and $g_j \in W_j$ can be periodized in the same manner, and we obtain

$$\tilde{f}_j := \sum_{n=-\infty}^{\infty} f_j(\cdot + nN) = \sum_{l=0}^{N_j-1} \tilde{\boldsymbol{c}}_{j,l}^{\mathrm{T}} \, \tilde{\boldsymbol{N}}_{m,j}^r(\cdot - 2^{-j} l), \tag{4.8}$$

$$\tilde{g}_j := \sum_{n=-\infty}^{\infty} g_j(\cdot + nN) = \sum_{l=0}^{N_j-1} \tilde{\boldsymbol{d}}_{j,l}^{\mathrm{T}} \, \tilde{\boldsymbol{\psi}}_{m,j}^r(\cdot - 2^{-j} l) \tag{4.9}$$

with

$$\tilde{\boldsymbol{c}}_{j,l} := \sum_{n=-\infty}^{\infty} \boldsymbol{c}_{j,l+nN_j}, \quad \tilde{\boldsymbol{d}}_j := \sum_{n=-\infty}^{\infty} \boldsymbol{d}_{j,l+nN_j} \quad (l = 0, \ldots, N_j - 1)$$

and

$$(4.10)\quad \tilde{\boldsymbol{N}}^r_{m,j} := \sum_{n=-\infty}^{\infty} \boldsymbol{N}^r_m(2^j\cdot - nN_j), \quad \tilde{\boldsymbol{\psi}}^r_{m,j} := \sum_{n=-\infty}^{\infty} \boldsymbol{\psi}^r_m(2^j\cdot - nN_j).$$

Observe that the DFT(N_{j+1})–data of $(\tilde{\boldsymbol{c}}_{j+1,l})_{l=0}^{N_{j+1}-1}$ read

$$(4.11)\qquad \boldsymbol{C}_{j+1}(w^k_{j+1}) = \sum_{n=0}^{N_{j+1}-1} \tilde{\boldsymbol{c}}_{j+1,n}\, w^{nk}_{j+1} \quad (k = 0,\ldots,N_{j+1}-1)$$

and analogously the DFT(N_j)–data of $(\tilde{\boldsymbol{c}}_{j,l})_{l=0}^{N_j-1}$, $(\tilde{\boldsymbol{d}}_{j,l})_{l=0}^{N_j-1}$

$$(4.12)\qquad \begin{aligned} \boldsymbol{C}_j(w^k_j) &= \sum_{n=0}^{N_j-1} \tilde{\boldsymbol{c}}_{j,n}\, w^{nk}_j, \\ \boldsymbol{D}_j(w^k_j) &= \sum_{n=0}^{N_j-1} \tilde{\boldsymbol{d}}_{j,n}\, w^{nk}_j \quad (k = 0,\ldots,N_j-1) \end{aligned}$$

with $w_j := \exp(-2\pi i/N_j)$. The decomposition and the reconstruction algorithm for $\tilde{f}_{j+1}$, $\tilde{f}_j$, $\tilde{g}_j$ is based on the following

Theorem 4.2. *For $j \in I\!N_0$, let $f_{j+1} \in V_{j+1}$, $f_j \in V_j$, $g_j \in W_j$ with* (4.1) *and* (4.3) *be given, and let $\tilde{f}_{j+1}$, $\tilde{f}_j$, $\tilde{g}_j$ their N-periodizations. Then the relations*

$$(4.13)\quad \begin{pmatrix} \boldsymbol{C}_{j+1}(w^k_{j+1}) \\ \boldsymbol{C}_{j+1}(-w^k_{j+1}) \end{pmatrix}^{\mathrm{T}} = 2\begin{pmatrix} \boldsymbol{C}_j(w^k_j) \\ \boldsymbol{D}_j(w^k_j) \end{pmatrix}^{\mathrm{T}} \boldsymbol{S}^r_m(w^k_{j+1}) \quad (k = 0,\ldots,N_j-1)$$

for the DFT- *data* (4.11) – (4.12) *imply that*

$$\tilde{f}_{j+1} = \tilde{f}_j + \tilde{g}_j.$$

Proof. Using Theorem 4.1, the relation (4.13) leads to

$$\hat{f}_{j+1}(2\pi u/N) = \hat{f}_j(2\pi u/N) + \hat{g}_j(2\pi u/N) \quad (u \in I\!R).$$

Hence, observing that the Fourier coefficients of an N-periodic function

$$\tilde{h} = \sum_{n=-\infty}^{\infty} h(\cdot + nN) \quad (h \in L^2(I\!R))$$

read

$$c_u(\tilde{h}) := \frac{1}{N}\int_0^N \tilde{h}(t)\, e^{-2\pi i u t/N}\ \mathrm{d}t = \frac{1}{N}\,\hat{h}(2\pi u/N) \quad (u \in Z\!\!Z),$$

we have

$$c_u(\tilde{f}_{j+1}) = c_u(\tilde{f}_j) + c_u(\tilde{g}_j) \quad (u \in Z\!\!Z).$$

Thus, $\tilde{f}_{j+1} = \tilde{f}_j + \tilde{g}_j$. □

From Theorem 4.2, we obtain immediately:

Algorithm 4.3. **(Decomposition algorithm)**
Input: $j \in IN_0$, $N \in IN$ (power of 2),
$N_j := 2^j N$,
$\boldsymbol{C}_{j+1}(w_{j+1}^k) \in \mathbb{C}^r \quad (k = 0, \dots N_{j+1} - 1)$.

Step 1: Precompute $\boldsymbol{S}_m^r(w_{j+1}^k)^{-1}$ $(k = 0, \dots, N_j - 1)$ given by (3.14) by FFT.

Step 2: Compute for $k = 0, \dots, N_j - 1$

$$\begin{pmatrix} \boldsymbol{C}_j(w_j^k) \\ \boldsymbol{D}_j(w_j^k) \end{pmatrix}^{\mathrm{T}} = \frac{1}{2} \begin{pmatrix} \boldsymbol{C}_{j+1}(w_{j+1}^k) \\ \boldsymbol{C}_{j+1}(-w_{j+1}^k) \end{pmatrix}^{\mathrm{T}} \boldsymbol{S}_m^r(w_{j+1}^k)^{-1}.$$

Output: $\boldsymbol{C}_j(w_j^k)$, $\boldsymbol{D}_j(w_j^k) \in \mathbb{C}^r \quad (k = 0, \dots, N_j - 1)$.

Algorithm 4.4. **(Reconstruction algorithm)**
Input: $j \in IN_0$, $N \in IN$ (power of 2),
$N_j := 2^j N$,
$\boldsymbol{C}_j(w_j^k)$, $\boldsymbol{D}_j(w_j^k) \in \mathbb{C}^r \quad (k = 0, \dots N_j - 1)$.

Step 1: Precompute $\boldsymbol{S}_m^r(w_{j+1}^k)$ $(k = 0, \dots, N_j - 1)$ given by (3.12) by FFT.

Step 2: Compute (4.13) for $k = 0, \dots, N_j - 1$.

Output: $\boldsymbol{C}_{j+1}(w_{j+1}^k) \in \mathbb{C}^r \quad (k = 0, \dots, N_{j+1} - 1)$.

Remark. If f_{j+1} is supported in $[-N/2, N/2]$, then there is no periodization error. Observe that the Algorithms 4.3 and 4.4 work exactly in the periodic case, using fast Fourier transform. They are not based on truncated Fourier sums. ♠

Now we are interested in the problem, how to get the needed input of the decomposition algorithm, and how to handle the output of decomposition and reconstruction algorithms above efficiently. In general, a given function $f \in L^1(IR)$ is not contained in V_j. We want to construct an approximation of f in V_j and, at the same time, find an efficient method to compute the needed input for the decomposition Algorithm 4.3. For convenience, we consider the N-periodization of $f \in L^2(IR)$

$$\tilde{f} = \sum_{n=-\infty}^{\infty} f(\cdot + nN). \tag{4.14}$$

The period $N \in IN$ is choosen as a power of 2 such that $\tilde{f}$ is a good approximation of f in $[-N/2,\ N/2]$. We want to find an N-periodic spline interpolant s of $\tilde{f}$

$$s = \sum_{l=0}^{N_j-1} \tilde{\boldsymbol{c}}_{j,l}^{\mathrm{T}} \, \tilde{\boldsymbol{N}}_{m,j}^r(\cdot - 2^{-j} l) \tag{4.15}$$

with $\tilde{\boldsymbol{N}}^r_{m,j}$ defined in (4.10) and $N_j := 2^j N$. Indeed, we are especially interested in the vectors $(\boldsymbol{c}_{j,l})_{l=0}^{N_j-1}$ ($\boldsymbol{c}_{j,l} := (c^\nu_{j,l})_{\nu=0}^{r-1}$) or in the Fourier transformed data $\boldsymbol{C}_j(w_j^k)$ ($k = 0, \ldots, N_j - 1$) defined in (4.12). These values can directly be employed in the decomposition Algorithm 4.3. Vice versa, if the Fourier transformed coefficients $\boldsymbol{C}_j(w_j^k)$, $\boldsymbol{D}_j(w_j^k)$ ($k = 0, \ldots, N_j - 1$) are known, computed by the decomposition– and reconstruction algorithm, respectively, then an algorithm is needed for the efficient computation of function values of $\tilde{f}_j$, $\tilde{g}_j$. Here again $\tilde{f}_j$, $\tilde{g}_j$ denote the N-periodization of $f_j \in V_j$ and $g_j \in W_j$.

In the following we want to present an efficient algorithm for the computation of $\boldsymbol{C}_j(w_j^k)$ ($k = 0, \ldots, N_j-1$). Let $m \in I\!N$ be odd and let $1 \le r \le \lfloor (m+1)/2 \rfloor$ be fixed. Consider the following N-periodic Hermite spline interpolation problem: For given values

$$\tilde{f}^\nu_{j,n} := \mathrm{D}^\nu \tilde{f}(2^{-j}x)|_{x=n} \quad (\nu = 0, \ldots, r-1;\ n = 0, \ldots, N_j - 1) \tag{4.16}$$

we wish to find an N-periodic spline function s satisfying the interpolation conditions

$$\mathrm{D}^\nu s(2^{-j}x)|_{x=n} = \tilde{f}^\nu_{j,n} \quad (\nu = 0, \ldots, r-1;\ n = 0, \ldots, N_j - 1). \tag{4.17}$$

Inserting (4.17) into (4.15), we find for $n = 0, \ldots, N_j - 1$

$$\tilde{\boldsymbol{f}}^{\mathrm{T}}_{j,n} = \sum_{l=0}^{N_j-1} \tilde{\boldsymbol{c}}^{\mathrm{T}}_{j,l} \left(\mathrm{D}^\nu \tilde{\boldsymbol{N}}^r_{m,j}(2^{-j}(x-l))|_{x=n} \right)_{\nu=0}^{r-1\ \mathrm{T}}$$

with $\tilde{\boldsymbol{f}}_{j,n} := (\tilde{f}^\nu_{j,n})_{\nu=0}^{r-1}$. Using the discrete Fourier transform of length N_j, we obtain

$$\hat{\boldsymbol{f}}^{\mathrm{T}}_{j,k} = \boldsymbol{C}(w_j^k)^{\mathrm{T}}\, \boldsymbol{H}^r_m(w_j^k)^{\mathrm{T}}$$

with

$$\hat{\boldsymbol{f}}_{j,k} := \sum_{n=0}^{N_j-1} \tilde{\boldsymbol{f}}_{j,n}\, w_j^{kn} \tag{4.18}$$

and the Euler–Frobenius matrix $\boldsymbol{H}^r_m$ of degree m and defect r defined as in (2.24), since

$$\begin{aligned}
&\sum_{n=0}^{N_j-1} \left(\mathrm{D}^\nu\, \tilde{\boldsymbol{N}}^r_{m,j}(2^{-j}x)|_{x=n} \right)_{\nu=0}^{r-1\ \mathrm{T}}\ w_j^{kn} \\
&= \sum_{u=-\infty}^{\infty} \sum_{n=0}^{N_j-1} (\mathrm{D}^\nu\, \boldsymbol{N}^r_m(x+uN_j)|_{x=n})_{\nu=0}^{r-1\ \mathrm{T}}\ w_j^{kn} \\
&= \sum_{s=-\infty}^{\infty} (\mathrm{D}^\nu\, \boldsymbol{N}^r_m(x)|_{x=s})_{\nu=0}^{r-1\ \mathrm{T}}\ w_j^{sk} = \boldsymbol{H}^r_m(w_j^k)^{\mathrm{T}}.
\end{aligned}$$

For odd m, the Euler–Frobenius matrix $\boldsymbol{H}_m^r$ is invertible (cf. [18, 23]). Thus, the Fourier transformed data are uniquely determined by

$$\boldsymbol{C}_j(w_j^k) = \boldsymbol{H}_m^r(w_j^k)^{-1}\,\hat{\boldsymbol{f}}_{j,k} \quad (k = 0, \ldots, N_j - 1). \tag{4.19}$$

We summarize:

Algorithm 4.5.

Input: $m,\, j \in I\!N$, m odd, $1 \le r \le \lfloor (m+1)/2 \rfloor$,
N power of 2, $N_j := 2^j N$,
$\tilde{f}_{j,n}^{\nu} \in \mathbb{C} \quad (\nu = 0, \ldots, r-1,\ n = 0, \ldots N_j - 1)$.

Step 1: Precompute $\boldsymbol{H}_m^r(w_j^k)^{-1}$ $(k = 0, \ldots, N_j - 1)$ given by (2.24).

Step 2: Compute for $\hat{\boldsymbol{f}}_{j,k}$ $(k = 0, \ldots, N_j - 1)$ in (4.18) by FFT.

Step 3: Compute (4.19).

Output: $\boldsymbol{C}_j(w_j^k) \in \mathbb{C}^r \quad (k = 0, \ldots, N_j - 1)$.

Remark. For small m and r, the matrix $\boldsymbol{H}_m^r(z)^{-1}$ $(z \in \mathbb{T})$ can be computed simply (see Example 5.1 for $\boldsymbol{H}_7^2(z)$). Further, we have

$$\boldsymbol{H}_3^2(z)^{-1} = \frac{1}{3z}\begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix},$$

$$\boldsymbol{H}_5^2(z)^{-1} = \frac{4}{5z\,(1 - 6z + z^2)}\begin{pmatrix} 5 - 10z & -1 - 5z \\ -10 + 5z & 5 + z \end{pmatrix}. \quad \spadesuit$$

For more information on Hermite spline interpolation and Euler–Frobenius matrices we refer to [24] and the references there. For even m, the Euler–Frobenius matrix is not invertible. However, using interpolation conditions with shifted interpolation nodes, i.e.,

$$\mathrm{D}^\nu s(2^{-j}x)|_{x=n+\tau} = \mathrm{D}^\nu \tilde{f}(2^{-j}x)|_{x=n+\tau} \quad (n = 0, \ldots, N_j - 1,\ \nu = 0, \ldots, r-1)$$

with $\tau \in (0,1)$, interpolating functions can be found analogously. For a good choice of τ see [24]. For $\lfloor (m+1)/2 \rfloor \le r \le m+1$ the Hermite spline interpolation problem can be solved locally.

Now we want to deal with the second problem, namely, how to compute approximate function values of $\tilde{f}_j$, $\tilde{g}_j$ efficiently, if the data $\boldsymbol{C}_j(w_j^k)$, $\boldsymbol{D}_j(w_j^k)$ $(k = 0, \ldots, N_j - 1)$ are known. Let us consider $\tilde{f}_j$ of the form (4.8) with $f \in V_j$. Its Fourier coefficients read

$$\begin{aligned} c_n(\tilde{f}_j) &= \frac{1}{N}\int_0^N \tilde{f}_j(x)\, e^{-2\pi i n x/N}\, \mathrm{d}x \\ &= \frac{1}{N}\,\boldsymbol{C}_j(w_j^n)^{\mathrm{T}} \int_0^N \tilde{\boldsymbol{N}}_{m,j}^r(x)\, e^{-2\pi i n x/N}\, \mathrm{d}x = \frac{1}{N_j}\boldsymbol{C}_j(w_j^n)^{\mathrm{T}}\, \hat{\boldsymbol{N}}_m^r(2\pi n/N_j). \end{aligned}$$

In order to compute $\tilde{f}_j$ we consider the Fourier series

$$\tilde{f}_j = \sum_{n=-\infty}^{\infty} c_n(\tilde{f}_j)\, e^{2\pi i n \cdot /N}.$$

We replace this series by the truncated sum

$$\sum_{l \in G_{\mu N}} c_l(\tilde{f}_j)\, e^{2\pi i l \cdot /N}$$

with $G_{\mu N} := \{l \in \mathbb{Z};\ -\mu N/2 < l \le \mu N/2\}$, where μ is a power of 2. Then, instead of the exact values $\tilde{f}_j(n/\mu)$ $(n = 0, \ldots, \mu N - 1)$ we calculate the approximate values

$$\tilde{f}_{j,n/\mu} := \sum_{l \in G_{\mu N}} c_l(\tilde{f}_j)\, w_{\mu N}^{-nl} \quad (n = 0, \ldots, \mu N - 1)$$

with $w_{\mu N} := \exp(-2\pi i/\mu N)$. We summarize:

Algorithm 4.6.
Input: $m,\ j \in \mathbb{N}$, $1 \le r \le m+1$,
N, ν powers of 2, $N_j := 2^j N$,
$\boldsymbol{C}_j(w_j^n)$ $(n = 0, \ldots, N_j - 1)$,
$\hat{\boldsymbol{N}}^r_m(2\pi l/N_j)$ $(l \in G_{\mu N})$ given by (2.9).

Step 1: Compute for $l \in G_{\mu N}$

$$c_l := \frac{1}{N_j}\, \boldsymbol{C}_j(w_j^{l'})^{\mathrm{T}}\, \hat{\boldsymbol{N}}^r_m(2\pi l/N_j)$$

with $l' := l \bmod N_j$.

Step 2: Put

$$\tilde{c}_k := \begin{cases} c_k & k = 0, \ldots, \mu N/2, \\ c_{k-\mu N} & k = \mu N/2, \ldots, \mu N - 1. \end{cases}$$

Step 3: Compute by FFT

$$\tilde{f}_{j,n/\mu} := \sum_{k=0}^{\mu N-1} \tilde{c}_k\, w_{\mu N}^{-kn} \quad (n = 0, \ldots, \mu N - 1).$$

Output: $\tilde{f}_{j,n/\mu} \in \mathbb{C}$ $\quad (n = 0, \ldots, \mu N - 1)$.

The computation of the approximation values $\tilde{g}_j$ in (4.9) with $g_j \in W_j$ can be made in the same manner. For the Fourier coefficients of g_j we find by the two–scale relation (3.3)

$$\begin{aligned} c_n(\tilde{g}_j) &= \frac{1}{N_j}\, \boldsymbol{D}_j(w_j^n)^{\mathrm{T}}\, \hat{\boldsymbol{\psi}}^r_m(2\pi n/N_j) \\ &= \frac{1}{N_j}\, \boldsymbol{D}_j(w_j^n)^{\mathrm{T}}\, \boldsymbol{Q}^r_m(w_{j+1}^n)\, \hat{\boldsymbol{N}}^r_m(\pi n/N_j) \quad (n \in \mathbb{Z}). \end{aligned}$$

Hence, we obtain the approximate values of $\tilde{g}_j(n/\mu)$

$$\tilde{g}_{j,n/\mu} := \sum_{l \in G_{\mu N}} c_l(\tilde{g}_j)\, w_{\mu N}^{-ln} \quad (n = 0, \ldots, \mu N - 1).$$

Remark. The algorithms presented in this section are closely related to the corresponding algorithms in [25]. ♠

5. Wavelets derived from Hermite fundamental splines

Let $m \in I\!N_0$ and $1 \le r \le m+1$ be fixed. In the following, we use the conditions (3.24) – (3.25) and put

$$\tilde{\boldsymbol{K}}(z^2) := 2^{-1}\, \boldsymbol{I}, \tag{5.1}$$

such that

$$\boldsymbol{Q}_m^r(z) = \frac{z}{2}\, (\boldsymbol{H}_{2m+1}^r(z)^{\mathrm{T}})^{-1}\, \boldsymbol{D}_{m,1}^r(z). \tag{5.2}$$

The corresponding wavelet vector $\boldsymbol{\psi}_m^r$, determined by (3.3), can be interpreted as follows (cf. [22]): Let $\hat{\boldsymbol{L}}_{2m+1}^r := (\hat{L}_\nu^{2m+1,r})_{\nu=0}^{r-1}$ be the Fourier transformed vector of spline functions $L_\nu^{2m+1,r} \in V_0^{2m+1,r}$ $(\nu = 0, \ldots, r-1)$ defined by

$$\hat{\boldsymbol{L}}_{2m+1}^r(u) := (\boldsymbol{H}_{2m+1}^r(e^{-iu})^{\mathrm{T}})^{-1}\, \hat{\boldsymbol{N}}_{2m+1}^r(u) \tag{5.3}$$

and $\boldsymbol{L}_{2m+1}^r = (L_\nu^{2m+1,r})_{\nu=0}^{r-1}$. Then we have for $n \in Z\!\!\!Z$

$$(\mathrm{D}^\mu\, L_\nu^{2m+1,r})(x)|_{x=n} = \delta_{0,n}\, \delta_{\mu,\nu} \quad (\mu,\, \nu = 0, \ldots, r-1)$$

with the Kronecker symbol δ (cf. [22]). The functions $L_\nu^{2m+1,r}$ are called *cardinal Hermite fundamental splines.* Now let

$$\psi_\nu^{m,r} := (\mathrm{D}^{m+1}\, L_\nu^{2m+1,r})(2\cdot -1) \quad (\nu = 0, \ldots, r-1) \tag{5.4}$$

and $\boldsymbol{\psi}_m^r := (\psi_\nu^{m,r})_{\nu=0}^{r-1}$. Using the relation (5.3), we obtain for the Fourier transformed vector $\hat{\boldsymbol{\psi}}_m^r$

$$\begin{aligned}
\hat{\boldsymbol{\psi}}_m^r(u) &= [(\mathrm{D}^{m+1}\, \boldsymbol{L}_{2m+1}^r)(2\cdot -1)]^\wedge(u) \\
&= 2^{-1}\, (iu/2)^{m+1}\, e^{-iu/2}\, \hat{\boldsymbol{L}}_{2m+1}^r(u/2) \\
&= 2^{-1}\, e^{-iu/2}\, (\boldsymbol{H}_{2m+1}^r(e^{-iu/2})^{\mathrm{T}})^{-1}\, \boldsymbol{D}_{m,1}^r(e^{-iu/2})\, \hat{\boldsymbol{N}}_m^r(u/2).
\end{aligned}$$

Thus, the vector of wavelets $\boldsymbol{\psi}_m^r$, determined by the two–scale symbol in (5.2) is the $(m+1)$-th derivative of the Hermite fundamental spline vector $\boldsymbol{L}_{2m+1}^r(2\cdot -1)$.

Remark. For $r = 1$, the two–scale symbol reads

$$\boldsymbol{Q}_m^1(z) = 2^{-1}\, z(1-z)^{m+1}\, (\boldsymbol{H}_{2m+1}^1(z))^{-1} = 2^{-1}\, z^{-m}\, (1-z)^{m+1}\, (\boldsymbol{\Phi}_m^1(z))^{-1}.$$

Hence, $\boldsymbol{Q}_m^1$ is the two–scale symbol of the interpolatory spline wavelet described in [7], pp. 177 – 182. ♠

Example 5.1. Let us consider the case of cubic spline wavelets of defect 2. From Example 2.1, 2. it follows for $z \in \boldsymbol{T}$ by

$$\boldsymbol{H}_7^2(z)^{-1} = \frac{12}{7z\,\Delta_7^2(z)} \begin{pmatrix} 21 + 224z - 175z^2 & -3 - 176z - 37z^2 \\ -175 + 224z + 21z^2 & 37 + 176z + 3z^2 \end{pmatrix}$$

with

$$\Delta_7^2(z) = 1 - 72z + 262z^2 - 72z^3 + z^4$$

for the two–scale symbol

$$\boldsymbol{Q}_3^2(z) = \frac{60}{\Delta_7^2(z)} \begin{pmatrix} \begin{array}{c} 7(1 + 40z + 30z^2 \\ -64z^3 - 7z^4) \end{array} & \begin{array}{c} 7(-7 - 64z + 30z^2 \\ +40z^3 + z^4) \end{array} \\ & \\ \begin{array}{c} -(1 + 100z + 478z^2 \\ +252z^3 + 9z^4) \end{array} & \begin{array}{c} 9 + 252z + 478z^2 \\ +100z^3 + z^4 \end{array} \end{pmatrix}. \quad \spadesuit$$

Fig. 1. Cubic spline wavelet $\psi_0^{3,2}$ defined by (5.4)

In the matrix $\boldsymbol{Q}_3^2(z)$, a certain symmetry can be observed. For $r = 2$, we can prove the following symmetry relations for $\psi_\nu^{m,r}$ $(\nu = 0, \ldots, r-1)$.

Theorem 5.1. *Let $r, m \in I\!N$ $(1 \le r \le m+1)$ be fixed. Then the wavelets $\psi_\nu^{m,r}$ $(\nu = 0, \ldots, r-1)$ determined by the two–scale symbol $\boldsymbol{Q}_m^r$ in (5.2), satisfy the symmetry relations*

$$\psi_\nu^{m,r}(1/2 + \cdot) = (-1)^{m+1-\nu}\, \psi_\nu^{m,r}(1/2 - \cdot). \tag{5.5}$$

Fig. 2. Cubic spline wavelet $\psi_1^{3,2}$ defined by (5.4)

Proof. Using the symmetry of the cardinal Hermite fundamental splines defined in (5.3)

$$L_\nu^{2m+1,r}(x) = (-1)^\nu \, L_\nu^{2m+1,r}(-x) \quad (\nu = 0, \ldots, r-1),$$

we obtain for $\nu = 0, \ldots, r-1$

$$\begin{aligned} \psi_\nu^{m,r}(y/2+1/2) &= 2^{m+1}\, \mathrm{D}^{m+1} L_\nu^{2m+1,r}(y) \\ &= (-1)^\nu 2^{m+1}\, \mathrm{D}^{m+1}[L_\nu^{2m+1,r}(-y)] \\ &= (-1)^{m+1-\nu}\, \psi_\nu^{m,r}(1/2 - y/2). \end{aligned}$$

□

6. Spline wavelets with minimal support

Let $\boldsymbol{Q}_m^r$ ($m \in I\!N_0$, $1 \leq r \leq m+1$) be of the form (3.24). We want to compute the matrix $\tilde{\boldsymbol{K}}(z^2)$ such that the corresponding wavelet vector $\boldsymbol{\psi}_m^r$, determined by

$$\boldsymbol{Q}_m^r(z) = z\, \tilde{\boldsymbol{K}}(z^2)\, (\boldsymbol{H}_{2m+1}^r(z)^{\mathrm{T}})^{-1}\, \boldsymbol{D}_{m,1}^r(z)$$

possesses minimal support beginning at zero.
Observe that $\boldsymbol{D}_{m,1}^r(z)$ is a matrix polynomial of the form $\sum_{l=0}^{m+1} \boldsymbol{D}_{m,1,l}^r\, z^l$, where $\boldsymbol{D}_{m,1,l}^r \in \mathbb{C}^{r\times r}$ and $\boldsymbol{D}_{m,1,0} \neq \boldsymbol{0}$. Since the elements of $(\boldsymbol{H}_{2m+1}^r(z)^{\mathrm{T}})^{-1}$ are Laurent series in the Wiener class, we have to look for a matrix polynomial $\boldsymbol{C}_m^r(z)$ with minimal degree such that $\boldsymbol{C}_m^r(z)\, \boldsymbol{H}_{2m+1}^r(z)^{\mathrm{T}}$ is of the form

$$\boldsymbol{C}_m^r(z)\, \boldsymbol{H}_{2m+1}^r(z)^{\mathrm{T}} = z\tilde{\boldsymbol{K}}(z^2) \tag{6.1}$$

and with

$$0 < c_0 \le \sigma_\mu(\boldsymbol{C}_m^r(z)\, \boldsymbol{C}_m^r(z)^\star) \le c_1 < \infty \quad (\mu = 0, 1), \tag{6.2}$$

i.e.,

$$\boldsymbol{Q}_m^r(z) = \boldsymbol{C}_m^r(z)\, \boldsymbol{D}_{m,1}^r(z). \tag{6.3}$$

The relation (6.1) is equivalent to the condition

$$\boldsymbol{C}_m^r(z)\, \boldsymbol{H}_{2m+1}^r(z)^{\mathrm{T}} + \boldsymbol{C}_m^r(-z)\, \boldsymbol{H}_{2m+1}^r(-z)^{\mathrm{T}} = \boldsymbol{0}. \tag{6.4}$$

We show that the computation of $\boldsymbol{C}_m^r(z)$ with (6.2) and (6.4) can be done by solving the following Hermite spline interpolation problem (cf. [12]): We wish to find spline functions $s_\nu \in V_1^{2m+1,r}$ $(\nu = 0, \ldots, r-1)$ with support on $[x_\nu,\, x_{\nu+2m+2-r}]$ and integer knots at

$$x_\nu, \ldots, x_{\nu+2m+2-r} \tag{6.5}$$

such that $\boldsymbol{s} := (s_0, \ldots, s_{r-1})^{\mathrm{T}}$ satisfies the following interpolation conditions

$$\mathrm{D}^\mu \boldsymbol{s}(x)|_{x=n} = \boldsymbol{0} \quad (\mu = 0, \ldots, r-1;\ n \in \mathbb{Z}). \tag{6.6}$$

In [12], it is shown that the splines $s_\nu \in V_1^{2m+1,r}$ $(\nu = 0, \ldots, r-1)$ are uniquely determined by (6.5) – (6.6) up to normalization. Furthermore, there exist no nontrivial solutions s_ν $(\nu = 0, \ldots, r-1)$ of (6.6) in $V_1^{2m+1,r}$ with smaller support (cf. [12]). We put

$$\boldsymbol{s} = \sum_{l=0}^{N-1} \boldsymbol{C}_l^{m,r}\, \boldsymbol{N}_{2m+1}^r(2\cdot - l) \quad (\boldsymbol{C}_l^{m,r} \in \mathbb{C}^{r\times r}). \tag{6.7}$$

By (6.5) and supp $N_\nu^{2m+1,r} = [x_\nu,\, x_{\nu+2m+2}]$, we put $N := \lfloor (2m+1)/r \rfloor$ in (6.7). Then, $\boldsymbol{C}_0^{m,r}$ and $\boldsymbol{C}_{N-1}^{m,r}$ are upper- and lower-triangular matrices, respectively. Let

$$\boldsymbol{C}_m^r(z) := \frac{1}{2} \sum_{l=0}^{N-1} \boldsymbol{C}_l^{m,r}\, z^l \tag{6.8}$$

be the symbol matrix of $\boldsymbol{s}$. The Fourier transform of (6.7) reads

$$\hat{\boldsymbol{s}} = \boldsymbol{C}_m^r(e^{-i\cdot/2})\, \hat{\boldsymbol{N}}_{2m+1}^r(\cdot/2). \tag{6.9}$$

We obtain:

Theorem 6.1. *The matrix polynomial $\boldsymbol{C}_m^r$ in (6.8), defined by the solution $\boldsymbol{s} = (s_\nu)_{\nu=0}^{r-1}$ of the interpolation problem (6.5) – (6.6), is unique up to normalization and satisfies the conditions (6.2) and (6.4). Moreover, there is no matrix polynomial satisfying (6.2) and (6.4) with a smaller degree than $N-1$.*

Proof. 1. Let $\boldsymbol{C}_l^{m,r} := \boldsymbol{0}$ for $l < 0$ and $l > N-1$,

$$\boldsymbol{C}_0(z^2) = \boldsymbol{C}_{m,0}^r(z^2) := 2^{-1}(\boldsymbol{C}_m^r(z) + \boldsymbol{C}_m^r(-z)) = \sum_{l=-\infty}^{\infty} \boldsymbol{C}_{2l}^{m,r}\, z^{2l},$$

$$\boldsymbol{C}_1(z^2) = \boldsymbol{C}_{m,1}^r(z^2) := (2z)^{-1}(\boldsymbol{C}_m^r(z) - \boldsymbol{C}_m^r(-z)) = \sum_{l=-\infty}^{\infty} \boldsymbol{C}_{2l+1}^{m,r}\, z^{2l},$$

such that

$$\boldsymbol{C}_m^r(z) = \boldsymbol{C}_0(z^2) + z\,\boldsymbol{C}_1(z^2). \tag{6.10}$$

Analogously,

$$\begin{aligned}\boldsymbol{H}_0(z^2) = \boldsymbol{H}_{2m+1,0}^r(z^2) &:= 2^{-1}(\boldsymbol{H}_{2m+1}^r(z) + \boldsymbol{H}_{2m+1}^r(-z)),\\ \boldsymbol{H}_{-1}(z^2) = \boldsymbol{H}_{2m+1,-1}^r(z^2) &:= 2^{-1}\, z(\boldsymbol{H}_{2m+1}^r(z) - \boldsymbol{H}_{2m+1}^r(-z)),\end{aligned}$$

i.e.,

$$\boldsymbol{H}_{2m+1}^r(z) = \boldsymbol{H}_0(z^2) + z^{-1}\,\boldsymbol{H}_{-1}(z^2). \tag{6.11}$$

From the interpolation conditions (6.6) it follows by (6.7) that

$$\sum_{l=-\infty}^{\infty} \boldsymbol{C}_l^{m,r}\, \mathrm{D}^{\mu} \boldsymbol{N}_{2m+1}^r(2x-l)|_{x=n} = \boldsymbol{0} \quad (n \in \mathbb{Z},\ \mu = 0, \ldots, r-1).$$

Thus, we have for $\mu = 0, \ldots, r-1$

$$\begin{aligned}&\sum_{l=-\infty}^{\infty} \boldsymbol{C}_l^{m,r} \sum_{n=-\infty}^{\infty} \mathrm{D}^{\mu} \boldsymbol{N}_{2m+1}^r(2x-l)|_{x=n}\, z^{2n}\\ &= \sum_{l=-\infty}^{\infty} \boldsymbol{C}_{2l}^{m,r}\, z^{2l} \sum_{n=-\infty}^{\infty} \mathrm{D}^{\mu} \boldsymbol{N}_{2m+1}^r(2(x-l))|_{x=n}\, z^{2(n-l)}\\ &\quad + \sum_{l=-\infty}^{\infty} \boldsymbol{C}_{2l+1}^{m,r}\, z^{2l} \sum_{n=-\infty}^{\infty} \mathrm{D}^{\mu} \boldsymbol{N}_{2m+1}^r(2(x-l)-1)|_{x=n}\, z^{2(n-l)} = \boldsymbol{0}.\end{aligned}$$

By definition (2.25) of $\boldsymbol{H}_{2m+1}^r(z)^{\mathrm{T}}$ we obtain

$$\boldsymbol{C}_0(z^2)\, \boldsymbol{H}_0(z^2)^{\mathrm{T}} + \boldsymbol{C}_1(z^2)\, \boldsymbol{H}_{-1}(z^2)^{\mathrm{T}} = \boldsymbol{0}.$$

Using the relations (6.10) and (6.11) we find (6.4).
2. Since the functions $s_{\nu}(\cdot - l)$ ($\nu = 0, \ldots, r-1$; $l \in \mathbb{Z}$) form a Riesz basis of

$$U := \{f \in V_1^{2m+1,r} :\ \mathrm{D}^{\mu} f(x)|_{x=l} = 0,\ l \in \mathbb{Z},\ \mu = 0, \ldots, r-1\}$$

(cf. [12]), we obtain that for $u \in IR$

$$\begin{aligned}
[\hat{\boldsymbol{s}},\, \hat{\boldsymbol{s}}](u) &= \sum_{n=-\infty}^{\infty} \hat{\boldsymbol{s}}(u+2\pi n)\, \hat{\boldsymbol{s}}(u+2\pi n)^{\star} \\
&= \sum_{n=-\infty}^{\infty} \boldsymbol{C}_m^r(e^{-i(u/2+\pi n)})\, \hat{\boldsymbol{N}}^r_{2m+1}(u/2+\pi n) \\
&\quad \times \hat{\boldsymbol{N}}^r_{2m+1}(u/2+\pi n)^{\star}\, \boldsymbol{C}_m^r(e^{-i(u/2+\pi n)})^{\star} \\
&= \boldsymbol{C}_m^r(e^{-iu/2})\, \boldsymbol{\Phi}^r_{2m+1}(e^{-iu/2}) \boldsymbol{C}_m^r(e^{-iu/2})^{\star} \\
&\quad + \boldsymbol{C}_m^r(-e^{-iu/2})\, \boldsymbol{\Phi}^r_{2m+1}(-e^{-iu/2}) \boldsymbol{C}_m^r(-e^{-iu/2})^{\star}
\end{aligned}$$

is positive definite with eigenvalues bounded away from zero. Thus, there are constants $\tilde{c}_0$, $\tilde{c}_1$ with

$$0 < \tilde{c}_0 \le \sigma_\mu\Big(\boldsymbol{C}_m^r(z)\, \boldsymbol{C}_m^r(z)^{\star} + \boldsymbol{C}_m^r(-z)\, \boldsymbol{C}_m^r(-z)^{\star}\Big) \le \tilde{c}_1 < \infty \quad (\mu = 0,1;\ z \in \boldsymbol{T}).$$

But by(6.4), $\sigma_0(\boldsymbol{C}_m^r(z)\, \boldsymbol{C}_m^r(z)^{\star}) = 0$ would yield $\sigma_0(\boldsymbol{C}_m^r(-z)\, \boldsymbol{C}_m^r(-z)^{\star}) = 0$ $(z \in \boldsymbol{T})$. Hence, (6.2) is satisfied for some positive constants c_0, c_1.
Finally, there is no matrix polynomial satisfying (6.2) and (6.4) with a smaller degree than $N-1$, since there are no nontrivial solutions s_ν $(\nu = 0, \ldots, r-1)$ of (6.6) with smaller support than $[x_\nu, x_{\nu+2m+2-r}]$ (cf. [12]). □

Now let $\boldsymbol{C}_m^r(z)$ be defined by (6.8) as symbol matrix of $\boldsymbol{s}$. Observing that

$$\begin{aligned}
[\mathrm{D}^{m+1}\boldsymbol{s}]^\wedge(u) &= \boldsymbol{C}_m^r(e^{-iu/2})\, [\mathrm{D}^{m+1}\boldsymbol{N}^r_{2m+1}]^\wedge(u/2) \\
&= \boldsymbol{C}_m^r(e^{-iu/2})\, \boldsymbol{D}^r_{m,1}(e^{-iu/2})\, \hat{\boldsymbol{N}}^r_m(u/2) \\
&= \boldsymbol{Q}_m^r(e^{-iu/2})\, \hat{\boldsymbol{N}}^r_m(u/2),
\end{aligned}$$

we find that the wavelet vector $\boldsymbol{\psi}_m^r$, determined by the two–scale symbol $\boldsymbol{Q}_m^r(z)$ in (6.3), can be described by the $(m+1)$-th derivative of $\boldsymbol{s}$

$$\boldsymbol{\psi}_m^r = \mathrm{D}^{m+1}\boldsymbol{s}. \tag{6.12}$$

Remark. The orthogonality relation (3.9) for $\boldsymbol{Q}_m^r$ is equivalent to the interpolation condition (6.4), since by (2.14), (2.26) and (6.3) we have

$$\boldsymbol{P}_m^r(z)\, \boldsymbol{\Phi}_m^r(z)\, \boldsymbol{Q}_m^r(z)^{\star} = 2^{-m-1}\, \boldsymbol{D}_m^r(z^2)\, \boldsymbol{P}^r_{-1}\, \boldsymbol{D}_r\, \overline{\boldsymbol{H}^r_{2m+1}(z)}\, \boldsymbol{C}_m^r(z)^{\star}. \quad \spadesuit$$

Example 6.1. For $r = 1$, the condition (6.4) is satisfied with the polynomial $\boldsymbol{C}_m^1(z) = c_m^1\, z^{-1}\boldsymbol{H}^1_{2m+1}(-z)$ of degree $2m$. Thus, by $\boldsymbol{D}^1_{m,1}(z) = (1-z)^{m+1}$ and choosing the normalization constant $c_m^1 = 2^{-m-1}$, we find

$$\begin{aligned}
\boldsymbol{Q}_m^1(z) &= z^{-1}\left(\frac{1-z}{2}\right)^{m+1} \boldsymbol{H}^1_{2m+1}(-z) \\
&= \left(\frac{1-z}{2}\right)^{m+1} \sum_{n=0}^{2m} N_{2m+1}(n+1)\,(-z)^n,
\end{aligned}$$

i.e., the well-known Chui–Wang wavelet is obtained (cf. [9, 8]).
For $r = 2$, the following symbols $\boldsymbol{C}_m^2$ and $\boldsymbol{Q}_m^2$ are found:
$m = 1:$

$$\boldsymbol{C}_1^2(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{Q}_1^2(z) = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2+z & -3 \\ -3z & 1+2z \end{pmatrix},$$

$m = 2:$

$$\boldsymbol{C}_2^2(z) = \begin{pmatrix} 20+3z & -7 \\ -7z & 3+20z \end{pmatrix},$$

$$\boldsymbol{Q}_2^2(z) = \frac{1}{8\sqrt{10}} \begin{pmatrix} 20+66z+14z^2 & 29+20z+z^2 \\ -14-66z-20z^2 & 1+20z+29z^2 \end{pmatrix},$$

$m = 3:$

$$\boldsymbol{C}_3^2(z) = \begin{pmatrix} 1653+5896z+66z^2 & -2879-792z \\ -792z-2879z^2 & 66+5896z+1653z^2 \end{pmatrix},$$

$$\boldsymbol{Q}_3^2(z) = \frac{1}{8\sqrt{181830}} \begin{pmatrix} 551+7232z+12325z^2 & -2041-11512z \\ +1826z^3+11z^4 & -8062z^2-330z^3 \\ & \\ -330z-8062z^2 & 11+1826z+12325z^2 \\ -11512z^3-2041z^4 & +7232z^3+551z^4 \end{pmatrix}.$$

Here the normalization constants are chosen, such that $\det \boldsymbol{Q}_m^2(-1) = 1$. ♠

Fig. 3. Cubic spline wavelet $\psi_0^{3,2}$ of minimal support.

Fig. 4. Cubic spline wavelet $\psi_1^{3,2}$ of minimal support.

The support of the obtained wavelets is given by

$$(6.13)\quad \operatorname{supp}\, \psi_\nu^{m,r} = [0, \lfloor(\nu + 2m + 2 - r)/r\rfloor] = [0, \lfloor(\nu + 2m + 2)/r\rfloor - 1]$$

for $\nu = 0, \ldots, r-1$. For $r = 1$, it follows that supp $\psi_0^{m,1} = [0, 2m+1]$. For $r = 2$, we have supp $\psi_\nu^{m,2} = [0, m]$ $(\nu = 0, 1)$. In the following, we suppose that for all s_ν $(\nu = 0, \ldots, r-1)$ the same normalization constant is used. Then a symmetry relation can be observed for $r = 2$.

Lemma 6.2. *Let $m \in IN$, $r = 2$ be fixed. The wavelets $\psi_\nu^{m,2}$ $(\nu = 0, 1)$ with minimal support, determined by the two–scale symbol $\boldsymbol{Q}_m^2$ in (6.3) with $\boldsymbol{C}_m^2(z)$ in (6.8), satisfy the relation*

$$(6.14)\qquad \psi_0^{m,2} = (-1)^{m+1}\, \psi_1^{m,2}(m - \cdot).$$

Proof. It can simply be shown that the spline functions s_0, s_1, determined by (6.5) and (6.6), satisfy the symmetry relation

$$s_0 = s_1(m - \cdot).$$

By repeated differentiation, the assertion follows by (6.12). □

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