ESTIMATIONS OF LINEAR WIDTHS OF THE CLASSES $B^{\Omega}_{p,\theta}$ OF PERIODIC FUNCTIONS OF MANY VARIABLES IN THE SPACE L_q

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The exact-order estimates of linear widths are established for the classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables in the space L_q for certain relations between on the parameters p and q.

Introduction

In the present paper, we obtain the exact-order estimates for the linear widths of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables in the space L_q. In what follows, we consider these quantities in more detail. First, we present necessary notation and definitions.

Let $\mathbb{R}^d$, $d \geq 1$, be a d-dimensional Euclidean space with elements

$$\mathrm{x} = (x_1, \ldots, x_d), \quad \mathrm{y} = (y_1, \ldots, y_d), \quad (\mathrm{x}, \mathrm{y}) = x_1 y_1 + \ldots + x_d y_d$$

and let $L_p(\pi_d)$ be a space of functions $f(\mathrm{x}) = f(x_1, \ldots, x_d)$ 2π-periodic in each variable and summable to the power p for $1 \leq p < \infty$ and essentially bounded for $p = \infty$ in a cube

$$\pi_d = \prod_{j=1}^{d} [0, 2\pi].$$

The norm of the functions in this cube is defined as follows:

$$\|f\|_{L_p(\pi_d)} = \|f\|_p = \left((2\pi)^{-d} \int_{\pi_d} \left| f(\mathrm{x}) \right|^p d\mathrm{x} \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L_\infty(\pi_d)} = \|f\|_\infty = \operatorname*{ess\,sup}_{\mathrm{x} \in \pi_d} \left| f(\mathrm{x}) \right|.$$

Further, for the sake of convenience, we write L_p instead of $L_p(\pi_d)$.

For $f \in L_p$ and $\mathrm{h} \in \mathbb{R}^d$, we set

$$\Delta_{\mathrm{h}} f(\mathrm{x}) = f(\mathrm{x} + \mathrm{h}) - f(\mathrm{x}).$$

For a function f, we define its multiple difference of the order $l \in \mathbb{N}$ at the point $\mathrm{x} = (x_1, \ldots, x_d)$ with step h by the formula

$$\Delta^l_{\mathrm{h}} f(\mathrm{x}) = \Delta_{\mathrm{h}} \Delta^{l-1}_{\mathrm{h}} f(\mathrm{x}), \qquad \Delta^0_{\mathrm{h}} f(\mathrm{x}) = f(\mathrm{x}).$$

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This formula also admits the following representation:

$$\Delta_{\mathrm{h}}^{l} f(\mathrm{x}) = \sum_{n=0}^{l} (-1)^{l+n} C_l^n f(\mathrm{x} + n\mathrm{h}).$$

The modulus of continuity of a function $f \in L_p$ of order $l \in \mathbb{N}$ is defined by the formula

$$\Omega_l(f;t)_p = \sup_{|\mathrm{h}| \le t} \left\| \Delta_{\mathrm{h}}^{l} f(\cdot) \right\|_p,$$

where $|\mathrm{h}|$ is the Euclidean norm of h.

Let $\Omega(t)$ be a function of the type of modulus of continuity of order l, i.e., $\Omega(t)$ is defined on $\mathbb{R}_+ = \{t \colon t \ge 0\}$ and satisfies the following conditions:

(i) $\Omega(0) = 0$ and $\Omega(t) > 0$ for $t > 0$;

(ii) $\Omega(t)$ is continuous;

(iii) $\Omega(t)$ is nondecreasing;

(iv) for all $n \in \mathbb{Z}_+$, $\Omega(nt) \le C_1 n^l \Omega(t)$, where $l \in \mathbb{N}$ and $C_1 > 0$ is a constant independent of n and t.

By Ψ_l, we denote the set of these functions Ω. Note that if $f \in L_p$, then $\Omega_l(f,t)_p \in \Psi_l$.

We also assume that Ω belongs to the sets S^α and S_l. This means the following:

I. $\Omega \in S^\alpha$, $\alpha > 0$, if the function $\Omega(\tau)/\tau^\alpha$ almost increases, i.e., there exists a constant $C_2 > 0$ independent of τ_1 and τ_2 and such that

$$\frac{\Omega(\tau_1)}{\tau_1^\alpha} \le C_2 \frac{\Omega(\tau_2)}{\tau_2^\alpha}, \quad 0 < \tau_1 \le \tau_2.$$

II. $\Omega \in S_l$ if there exists γ, $0 < \gamma < l$, such that the function $\Omega(\tau)/\tau^\gamma$ almost decreases, i.e., there exists a constant $C_3 > 0$ independent of τ_1 and τ_2 and such that

$$\frac{\Omega(\tau_1)}{\tau_1^\gamma} \ge C_3 \frac{\Omega(\tau_2)}{\tau_2^\gamma}, \quad 0 < \tau_1 \le \tau_2.$$

The conditions under which the function Ω belongs to the sets S^α and S_l are called the Bari–Stechkin conditions [1].

We also set $\Phi_{\alpha,l} = \Psi_l \cap S^\alpha \cap S_l$.

For the sake of clarity, we now present the following example of a function $\Omega \in \Phi_{\alpha,l}$:

$$\Omega(t) = \begin{cases} t^r \left(\log^+ \left(\dfrac{1}{t} \right) \right)^b, & t > 0, \\ 0, & t = 0, \end{cases}$$

where $\log^+(\tau) = \max\{1, \log(\tau)\}$, $\alpha < r < l$, and b is a fixed real number.

We now directly proceed to the definition of the spaces $B_{p,\theta}^{\Omega}$ (see, e.g., [2]).

Let $1 \le p,\ \theta \le \infty$ and let $\Omega \in \Phi_{\alpha,l}$. Assume that $f \in B^{\Omega}_{p,\theta}$ if f satisfies the conditions

(i) $f \in L_p$;

(ii) $|f|_{B^{\Omega}_{p,\theta}} < \infty$,

where the seminorm $|f|_{B^{\Omega}_{p,\theta}}$ is defined by the relation

$$|f|_{B^{\Omega}_{p,\theta}} = \begin{cases} \left(\int\limits_0^{+\infty} \left(\dfrac{\Omega_l(f,t)_p}{\Omega(t)} \right)^{\theta} \dfrac{dt}{t} \right)^{1/\theta}, & 1 \le \theta < \infty, \\ \sup\limits_{t>0} \dfrac{\Omega_l(f,t)_p}{\Omega(t)}, & \theta = \infty. \end{cases}$$

The space $B^{\Omega}_{p,\theta}$ is linear normed space with the norm

$$\|f\|_{B^{\Omega}_{p,\theta}} = \|f\|_p + |f|_{B^{\Omega}_{p,\theta}}.$$

If $\Omega(t) = t^r$, then the spaces $B^{\Omega}_{p,\theta}$ coincide with the Besov spaces $B^r_{p,\theta}$ [3]. In particular, for $\theta = \infty$, we obtain $B^r_{p,\infty} = H^r_p$, where H^r_p are the spaces introduced by Nikol'skii [4]. If $\|f\|_{B^{\Omega}_{p,\theta}} \le 1$, then we say that the function f belongs to the class $B^{\Omega}_{p,\theta}$. In this case, we preserve for the classes the same notation as for the corresponding spaces $B^{\Omega}_{p,\theta}$.

Note that the following embeddings take place for the classes $B^{\Omega}_{p,\theta}$ with $1 < \theta < \theta' < \infty$:

$$B^{\Omega}_{p,1} \subset B^{\Omega}_{p,\theta} \subset B^{\Omega}_{p,\theta'} \subset B^{\Omega}_{p,\infty} \equiv H^{\Omega}_p. \tag{1}$$

Further, we assume that, for two nonnegative quantities A and B, the notation $A \asymp B$ means that there exists a constant $C_4 > 0$ such that

$$C_4^{-1} A \le B \le C_4 A.$$

The notation $A \ll B$ $(A \gg B)$ means that $C_4^{-1} A \le B$ $(B \le C_4 A)$. All constants $C_i,\ i \in \mathbb{N}$, in the present paper may depend solely on the parameters contained in the definitions of the class, metric in which the error of approximation is estimated, and the dimension of the space $\mathbb{R}^d$.

In what follows, for convenience, we use another definition of the classes $B^{\Omega}_{p,\theta}$.

By $V_m(t),\ m \in \mathbb{N},\ t \in \mathbb{R}$, we denote the de-la-Vallée-Poussin kernel

$$V_m(t) = 1 + 2\sum_{k=1}^{m} \cos kt + 2\sum_{k=m+1}^{2m-1} \left(\frac{2m-k}{m} \right) \cos kt.$$

We define the multidimensional kernel $V_m(\mathrm{x}),\ m \in \mathbb{N},\ \mathrm{x} \in \mathbb{R}^d$, by the formula

$$V_m(\mathrm{x}) = \prod_{j=1}^{d} V_m(x_j).$$

For a function $f \in L_p$, we consider the operator of convolution $\mathbf{V}_m$ of this function with the kernel V_m, i.e.,

$$\mathbf{V}_m f = f * V_m = \mathrm{V}_m(f, \mathrm{x}), \quad \mathrm{x} \in \mathbb{R}^d.$$

Thus, $V_m(f, \mathrm{x})$ is the multiple sum of the de-la-Vallée-Poussin function f. For $f \in L_p$, we set

$$\sigma_0(f, \mathrm{x}) = V_1(f, \mathrm{x}), \qquad \sigma_s(f, \mathrm{x}) = V_{2^s}(f, \mathrm{x}) - V_{2^{s-1}}(f, \mathrm{x}), \qquad s \in \mathbb{N}, \quad \mathrm{x} \in \mathbb{R}^d.$$

In terms of the introduced notation, for $1 \le p \le \infty$ (to within absolute constants), the classes $B^{\Omega}_{p,\theta}$ can be defined as follows (see, e.g., [2]):

$$B^{\Omega}_{p,\theta} = \big\{ f \in L_p \colon \|f\|_{B^{\Omega}_{p,\theta}} \le 1 \big\},$$

where

$$\|f\|_{B^{\Omega}_{p,\theta}} \asymp \begin{cases} \left(\sum_{s \in \mathbb{Z}_+} \left(\dfrac{\|\sigma_s(f, \cdot)\|_p}{\Omega(2^{-s})} \right)^{\theta} \right)^{1/\theta}, & 1 \le \theta < \infty, \\ \sup\limits_{s \in \mathbb{Z}_+} \dfrac{\|\sigma_s(f, \cdot)\|_p}{\Omega(2^{-s})}, & \theta = \infty. \end{cases} \tag{2}$$

It is worth noting that, in the case $1 < p < \infty$, the equivalent relations for the norms of functions from the classes $B^{\Omega}_{p,\theta}$, $1 \le \theta \le \infty$, can be written by using [in (2)] the binary "blocks" of the Fourier series for the function f instead of $\sigma_s(f, \mathrm{x})$.

We now present the definitions of the investigated approximate characteristics.

Let W be a centrally symmetric set in the Banach space $\mathcal{X}$. Then the linear width of the set W in the space $\mathcal{X}$ is defined by the relation

$$\lambda_m(W,\ \mathcal{X}) = \inf_{A} \sup_{x \in W} \|x - Ax\|_{\mathcal{X}},$$

where the infimum is taken over all linear operators A acting in $\mathcal{X}$ for which the dimension of the range of values do not exceed m. The notion of linear width was introduced by Tikhomirov in [5].

For the history of investigations in the field of linear widths of various classes of functions of many variables, see [6–10] and the references therein.

In finding the lower bounds of linear widths for the classes $B^{\Omega}_{p,\theta}$, we use the well-known estimates of their Kolmogorov widths. Recall that the Kolmogorov width of a centrally symmetric set W in the Banach space $\mathcal{X}$ is defined as follows [11]:

$$d_m(W,\ \mathcal{X}) = \inf_{L_m} \sup_{f \in W} \inf_{u \in L_m} \|f - u\|_{\mathcal{X}},$$

where L_m is a subspace of the space $\mathcal{X}$ whose dimension doe not exceed m.

It is easy to see that, according to the definition of linear and Kolmogorov widths, they satisfy the following inequality:

$$d_m(W,\ \mathcal{X}) \le \lambda_m(W,\ \mathcal{X}). \tag{3}$$

1. Auxiliary Statements

In the proofs of the main results, we use some well-known assertions reformulated by using the introduced notation.

By l_p^m, we denote the space of all possible ordered systems of m real numbers in which the norm is defined as follows:

$$\|x\|_{l_p^m} = \begin{cases} \left(\sum\limits_{i=1}^{m} |x_i|^p\right)^{1/p}, & 1 \le p < \infty, \\ \max\limits_{1\le i\le m} |x_i|, & p = \infty, \end{cases}$$

and B_p^m is the unit ball in this space.

Theorem A [12]. *Let* $m < M,\ 1 \le p < 2 \le q < \infty$, *and* $\frac{1}{p} + \frac{1}{q} \ge 1$. *Then*

$$\lambda_m\left(B_p^M,\ l_q^M\right) \asymp \max\left\{M^{\frac{1}{q}-\frac{1}{p}}, \min\left\{1, M^{1/q} m^{-\frac{1}{2}}\right\}\sqrt{1-\frac{m}{M}}\right\}.$$

Note that, for the case $p = 1,\ q > 2$, the corresponding result follows from the assertion established by Kashin [13] for the Kolmogorov width of an octahedron B_1^M in the space l_q^M.

For $s \in \mathbb{N}$, by $\rho(s)$, we denote a subset of the integer-valued lattice $\mathbb{Z}^d$ of the form

$$\rho(s) = \left\{\mathrm{k} = (k_1, \ldots, k_d)\colon 2^{s-1} \le |k_j| < 2^s,\ j = \overline{1, d}\right\}.$$

For $f \in L_1$, we set

$$\delta_s(f, \mathrm{x}) = \sum_{\mathrm{k}\in\rho(s)} \widehat{f}(\mathrm{k}) e^{i(\mathrm{k},\mathrm{x})},$$

where

$$\widehat{f}(\mathrm{k}) = (2\pi)^{-d} \int\limits_{\pi_d} f(\mathrm{t}) e^{-i(\mathrm{k},\mathrm{t})} d\mathrm{t}$$

are the Fourier coefficients of the function f.

By $\mathcal{T}(\rho(s))$, we denote a set of functions f of the form

$$f(\mathrm{x}) = \sum_{\mathrm{k}\in\rho(s)} c_{\mathrm{k}} e^{i(\mathrm{k},\mathrm{x})}.$$

Theorem B. *There exists an isomorphism between the space of trigonometric polynomials* $f \in \mathcal{T}\big(\rho(s)\big)$ *and the space* $\mathbb{R}^{2^{sd}}$. *This isomorphism associates the function* f *with a vector* $\delta_s f^j = \left\{f_{\mathrm{n}}(\tau_j)\right\} \in \mathbb{R}^{2^{sd}}$,

$$f_{\mathrm{n}}(\mathrm{t}) = \sum_{sgn k_l = n_l} c_{\mathrm{k}} e^{i(\mathrm{k},\mathrm{t})}, \qquad l = \overline{1, d}, \quad \mathrm{n} = (\pm1, \ldots, \pm1) \in \mathbb{R}^d,$$

$$\tau_j = \pi 2^{2-s}(j_1, \ldots, j_d), \qquad j_i = 1, 2, \ldots, 2^{s-1}, \quad i = \overline{1, d},$$

and, moreover, the relation

$$\big\|f(\cdot)\big\|_p \asymp 2^{-sd/p}\|\delta_s f^j\|_{l_p^{2^{sd}}}, \quad p\in(1,\infty),$$

is true.

The proof of the theorem is similar to the proof of the Marcinkiewicz–Zygmund theorem on discretization for a function of one variable [14, p. 46]. In the case of functions of many variables and for $\mathrm{s}=(s_1,\dots,s_d)$, this theorem is proved in [15].

Theorem C (Littlewood–Paley [16])**.** *Let $f\in L_p,\ 1<p<\infty$. Then there exist positive constants C_5 and C_6 such that*

$$C_5\|f\|_p \le \left\|\left(\sum_{s=0}^{\infty}\big|\delta_s(f,\cdot)\big|^2\right)^{1/2}\right\|_p \le C_6\|f\|_p.$$

By using the definition of a linear width, Theorem B, and the Littlewood–Paley theorem, we readily obtain the following statement:

Lemma A. *Let $s\in\mathbb{N}$ and let $f\in\mathcal{T}\big(\rho(s)\big),\ m_s\in\mathbb{Z}_+,\ m_s\le 2^{sd}$. If $1<p$ and $q<\infty$, then there exists a linear operator $\Lambda_{m_s}\colon\mathcal{T}\big(\rho(s)\big)\to\mathcal{T}\big(\rho(s)\big)$ for which the dimension of the range of values does not exceed m_s such that*

$$\|f-\Lambda_{m_s}f\|_q \asymp \lambda_{m_s}\big(B_p^{2^{sd}}, l_q^{2^{sd}}\big)2^{sd(1/p-1/q)}\|f\|_p. \tag{4}$$

A similar statement for the case $\mathrm{s}=(s_1,\dots,s_d),\ s_j\in\mathbb{N},\ j=\overline{1,d}$, can be found in [6].

Lemma B. *Let $1\le p<q<\infty$ and let $f\in L_p$. Then the relation*

$$\|f\|_q^q \ll \sum_s\Big(\|\delta_s(f,\cdot)\|_p 2^{sd(1/p-1/q)}\Big)^q$$

is true.

This lemma is proved by using an elementary modification of the reasoning applied by Temlyakov (see [17, p. 25]) in the proof of the corresponding lemma in the case $\mathrm{s}=(s_1,\dots,s_d)$.

Theorem D [4]**.** *Let $n_j\in\mathbb{N},\ j=\overline{1,d}$, and let*

$$t(\mathrm{x})=\sum_{|k_j|\le n_j} c_{\mathrm{k}}e^{i(\mathrm{k},\mathrm{x})}.$$

Then, for $1\le q<p\le\infty$, the inequality

$$\|t\|_p\le 2^d\prod_{j=1}^d n_j^{1/q-1/p}\|t\|_q \tag{5}$$

is true.

Inequality (5) is proved by Nikol'skii, and it is called "the inequality of different metrics." In the case $d=1$ and $p=\infty$, the corresponding inequality is established by Jackson [18].

2. Main Results

We now formulate and prove the main results of the present paper.

Theorem 1. *Let* $1 \le p < 2 \le q < p'$, $1 \le \theta \le \infty$, *and* $\Omega \in \Phi_{\alpha,l}$, $\alpha > d/p$. *Then the estimate*

$$\lambda_m(B^{\Omega}_{p,\theta}, L_q) \asymp \Omega(m^{-1/d}) m^{1/p-1/2}, \tag{6}$$

where $1/p + 1/p' = 1$, *is true.*

Proof. We first establish an upper bound of the quantity $\lambda_m(B^{\Omega}_{p,\theta}, L_q)$. According to embeddings (1), it suffices to obtain this upper bound for the class H^{Ω}_p. Consider the case $1 < p < 2 \le q < p'$.

For any $m \in \mathbb{N}$, we can select $n \in \mathbb{N}$ such that the relation $m \asymp 2^{nd}$ holds. We associate each $s \in \mathbb{Z}_+$ with numbers

$$m_s = \begin{cases} 2^{sd}, & 0 \le s \le n, \\ \left[2^{nd+\beta(nd-sd)}\right], & s > n, \end{cases}$$

where $\beta > 0$ is some number selected in what follows and $[a]$ is the integer part of the number a.

We now estimate $\sum_s m_s$ as follows:

$$\sum_s m_s \le \sum_{s=0}^{n} 2^{sd} + \sum_{s>n} 2^{nd+\beta(nd-sd)}$$

$$\ll 2^{nd} + 2^{nd+\beta nd} \sum_{s>n} 2^{-\beta sd} \ll 2^{nd} + 2^{nd} \asymp 2^{nd} \asymp m.$$

Let f be an arbitrary function from the class H^{Ω}_p. Consider the linear operator Λ_m of rank m that acts on f according to the relation

$$\Lambda_m f(\mathrm{x}) = \sum_s \Lambda_{m_s} \delta_s(f, \mathrm{x}),$$

where Λ_{m_s} are the operators constructed according to Lemma A.

Let us estimate $\|f(\cdot) - \Lambda_m f(\cdot)\|_q$. By successively using the Littlewood–Paley theorem, the Minkowski inequality, and relation (4), we obtain

$$\left\|f(\cdot) - \Lambda_m f(\cdot)\right\|_q \ll \left\| \left(\sum_{s>n} \left|\delta_s(f,\cdot) - \Lambda_{m_s}\delta_s(f,\cdot)\right|^2 \right)^{1/2} \right\|_q$$

$$= \left(\left\| \sum_{s>n} \left|\delta_s(f,\cdot) - \Lambda_{m_s}\delta_s(f,\cdot)\right|^2 \right\|_{q/2} \right)^{1/2}$$

$$\le \left(\sum_{s>n} \left\|\delta_s(f,\cdot) - \Lambda_{m_s}\delta_s(f,\cdot)\right\|_q^2 \right)^{1/2}$$

$$\asymp \left(\sum_{s>n} \lambda_{m_s}^2 \left(B_p^{2^{sd}}, l_q^{2^{sd}} \right) 2^{2sd(1/p-1/q)} \left\| \delta_s(f,\cdot) \right\|_p^2 \right)^{1/2} = \mathcal{I}_1.$$

According to Theorem A, we have

$$\lambda_{m_s} \left(B_p^{2^{sd}}, l_q^{2^{sd}} \right) \asymp \max \left\{ 2^{sd(1/q-1/p)}, \min \left\{ 1, 2^{sd/q} m_s^{-1/2} \right\} \sqrt{1 - \frac{m_s}{2^{sd}}} \right\}$$

$$\ll \max \left\{ 2^{sd(1/q-1/p)}, 2^{sd/q} m_s^{-1/2} \sqrt{1 - \frac{m_s}{2^{sd}}} \right\} \ll 2^{sd/q} m_s^{-1/2}.$$

We continue the estimate for $\mathcal{I}_1$ as follows:

$$\mathcal{I}_1 \ll \left(\sum_{s>n} 2^{2sd/q} m_s^{-1} 2^{2sd(1/p-1/q)} \left\| \delta_s(f,\cdot) \right\|_p^2 \right)^{1/2} = \left(\sum_{s>n} 2^{2sd/p} m_s^{-1} \left\| \delta_s(f,\cdot) \right\|_p^2 \right)^{1/2} = \mathcal{I}_2.$$

In $\mathcal{I}_2$, we replace m_s by their values. By taking into account the inequality

$$\left\| \delta_s(f,\cdot) \right\|_p \ll \Omega(2^{-s})$$

for the function $f \in H_p^{\Omega}$, we get

$$\mathcal{I}_2 \ll \left(\sum_{s>n} 2^{2sd/p} 2^{-nd-\beta(nd-sd)} \Omega^2(2^{-s}) \right)^{1/2}$$

$$= 2^{-nd/2-\beta nd/2} \left(\sum_{s>n} 2^{2sd/p} 2^{\beta sd} \frac{\Omega^2(2^{-s})}{2^{-2\alpha s}} 2^{-2\alpha s} \right)^{1/2}.$$

In view of the fact that $\Omega \in S^{\alpha},\ \alpha > d/p,$ we extend the estimate for the quantity $\mathcal{I}_2$ as follows:

$$\mathcal{I}_2 \ll 2^{-nd/2-\beta nd/2} \frac{\Omega(2^{-n})}{2^{-\alpha n}} \left(\sum_{s>n} 2^{2sd(1/p+\beta/2-\alpha/d)} \right)^{1/2} = \mathcal{I}_3.$$

Selecting $\beta > 0$ from the condition $1/p + \beta/2 - \alpha/d < 0$ (this is possible because $\alpha > d/p$), we obtain

$$\mathcal{I}_3 \ll 2^{-nd/2-\beta nd/2} \frac{\Omega(2^{-n})}{2^{-\alpha n}} 2^{nd(1/p+\beta/2-\alpha/d)}$$

$$= \Omega(2^{-n}) 2^{nd(1/p-1/2)} \asymp \Omega(m^{-1/d}) m^{1/p-1/2}.$$

Hence, we have established the upper bound for $\lambda_m(B_{p,\theta}^{\Omega}, L_q)$ in the case $1 < p < 2 \le q < p'$.

Now let $p=1$ and $2 \le q < \infty$. In this case, it is necessary to repeat the reasoning used in [19] to deduce the upper bound of the trigonometric width

$$d_m^T(H_1^\Omega, L_q), \quad 2 \le q < \infty.$$

To obtain the lower bound, we use inequality (3) and the well-known estimates for the Kolmogorov widths $d_m(B_{p,\theta}^\Omega, L_q)$ [20].

The theorem is proved.

Remark 1. In the case $d=1$, for $1<p<2\le q<p',\ 1\le\theta\le\infty$, and $p=1,\ 2<q<\infty,\ 1\le\theta\le\infty$ (here, $\alpha>1$), Theorem 1 was established in [21] and [22], respectively.

Theorem 2. *Let* $1<p\le 2,\ p'<q<\infty,\ 1\le\theta\le\infty$, *and let* $\Omega\in\Phi_{\alpha,l},\ \alpha>d(1-1/q)$. *Then*

$$\lambda_m(B_{p,\theta}^\Omega, L_q) \asymp \Omega(m^{-1/d}) m^{1/2-1/q}, \tag{7}$$

where $1/p+1/p'=1$.

Proof. We first establish the upper estimate of the quantity $\lambda_m(B_{p,\theta}^\Omega, L_q)$. By analogy with the previous theorem, it suffices to prove this theorem for the quantity $\lambda_m(H_p^\Omega, L_q)$.

Let $m\in\mathbb{N}$ and $n\in\mathbb{N}$ be such that $m\asymp 2^{nd}$. Further, assume that the numbers $m_s,\ s\in\mathbb{Z}_+$, and the operators Λ_m and Λ_{m_s} are the same as in Theorem 1.

Let us estimate $\|f-\Lambda_m f\|_q$. Under the condition of the theorem, $2\le p'<q<\infty$. Hence, by Lemma B, we get

$$\|f\|_q^q \ll \sum_s \left(\big\|\delta_s(f,\cdot)\big\|_{p'} 2^{sd(1/p'-1/q)}\right)^q.$$

By using this relation, we can write

$$\Big\|f(\cdot)-\Lambda_m f(\cdot)\Big\|_q \ll \left\|\sum_{s>n}\Big(\delta_s(f,\cdot)-\Lambda_{m_s}\delta_s(f,\cdot)\Big)\right\|_q$$

$$\ll \left(\sum_{s>n}\left(2^{sd(1/p'-1/q)}\big\|\delta_s(f,\cdot)-\Lambda_{m_s}\delta_s(f,\cdot)\big\|_{p'}\right)^q\right)^{1/q} = \mathcal{I}_4.$$

In view of relation (4), we extend the estimate for $\mathcal{I}_4$ as follows:

$$\mathcal{I}_4 \ll \left(\sum_{s>n}\left(2^{sd(1/p'-1/q)}\lambda_{m_s}\left(B_p^{2^{sd}}, l_{p'}^{2^{sd}}\right)2^{sd(1/p-1/p')}\big\|\delta_s(f,\cdot)\big\|_p\right)^q\right)^{1/q}$$

$$= \left(\sum_{s>n}\left(2^{sd(1/p-1/q)}\lambda_{m_s}\left(B_p^{2^{sd}}, l_{p'}^{2^{sd}}\right)\big\|\delta_s(f,\cdot)\big\|_p\right)^q\right)^{1/q} = \mathcal{I}_5.$$

Since, by Theorem A, the order estimate

$$\lambda_{m_s}\left(B_p^{2^{sd}}, l_{p'}^{2^{sd}}\right) \ll 2^{sd/p'} m_s^{-1/2}$$

is true, we find

$$\mathcal{I}_5 \ll \left(\sum_{s>n}\left(2^{sd(1/p-1/q)}2^{sd/p'}m_s^{-1/2}\big\|\delta_s(f,\cdot)\big\|_p\right)^q\right)^{1/q} = \left(\sum_{s>n}\left(2^{sd(1-1/q)}m_s^{-1/2}\big\|\delta_s(f,\cdot)\big\|_p\right)^q\right)^{1/q}.$$

Replacing in the last relation the quantities m_s by their values and taking into account the fact that $f \in H_p^{\Omega}$, we obtain

$$\mathcal{I}_5 \ll \left(\sum_{s>n}\left(2^{sd(1-1/q)}2^{-nd/2-\beta/2(nd-sd)}\Omega(2^{-s})\right)^q\right)^{1/q}$$

$$= 2^{-nd/2(1+\beta)}\left(\sum_{s>n}\left(2^{sd(1-1/q+\beta/2)}\Omega(2^{-s})\right)^q\right)^{1/q}$$

$$= 2^{-nd/2(1+\beta)}\left(\sum_{s>n}\left(2^{sd(1-1/q+\beta/2-\alpha/d)}\frac{\Omega(2^{-s})}{2^{-\alpha s}}\right)^q\right)^{1/q} = \mathcal{I}_6.$$

Note that, under the condition of the theorem, $\Omega \in S^{\alpha},\ \alpha > d(1-1/q)$. This enables us to conclude that

$$\frac{\Omega(2^{-s})}{2^{-\alpha s}} \ll \frac{\Omega(2^{-n})}{2^{-\alpha n}}, \quad s > n. \tag{8}$$

We select the number $\beta > 0$ from the condition

$$1 - 1/q + \beta/2 - \alpha/d < 0, \tag{9}$$

which is possible because $\alpha > d(1-1/q)$.

Substituting (8) in $\mathcal{I}_6$ and using (9), we get

$$\mathcal{I}_6 \ll 2^{-nd/2(1+\beta)}\frac{\Omega(2^{-n})}{2^{-\alpha n}}\left(\sum_{s>n}2^{qsd(1-1/q+\beta/2-\alpha/d)}\right)^{1/q}$$

$$\ll 2^{-nd/2(1+\beta)}\frac{\Omega(2^{-n})}{2^{-\alpha n}}2^{nd(1-1/q+\beta/2-\alpha/d)}$$

$$= \Omega(2^{-n})2^{nd(1/2-1/q)} \asymp \Omega(m^{-1/d})m^{1/2-1/q}.$$

By using the definition of linear width, we obtain the required upper bound.

We now establish the lower bound for $\lambda_m(B_{p,\theta}^{\Omega}, L_q)$. Since, for $1 < p \le 2$, we have $B_{p,\theta}^{\Omega} \supset B_{2,\theta}^{\Omega}$, in view of the embedding $B_{p,\theta}^{\Omega} \supset B_{p,1}^{\Omega}$ [see (1)], it suffices to establish the lower bound for the width $\lambda_m(B_{2,1}^{\Omega}, L_q)$.

We define $m \in \mathbb{N}$ and choose $l \in \mathbb{N}$ from the conditions $m \asymp 2^{ld}$ and $2^{ld} \ge 2m$. By $\mathcal{T}_l$, we denote the set of trigonometric polynomials with numbers of harmonics from $\rho(l)$. According to the definition of linear width, we find

$$\lambda_m(B^\Omega_{2,1}, L_q) \geq \lambda_m(B^\Omega_{2,1} \cap \mathcal{T}_l, L_q). \tag{10}$$

Further, let P_l be the operator of orthogonal projection onto the set $\mathcal{T}_l$. Then, for $f \in L_q$ and $t \in \mathcal{T}_l$, the relation

$$\|P_l f - t\|_q = \big\|P_l(f-t)\big\|_q \leq \|f-t\|_q \tag{11}$$

is true. In view of (10), we obtain the following relation from (11):

$$\lambda_m\big(B^\Omega_{2,1}, L_q\big) \geq \lambda_m\big(B^\Omega_{2,1} \cap \mathcal{T}_l, L_q \cap \mathcal{T}_l\big). \tag{12}$$

Now let $f \in L_2 \cap \mathcal{T}_l$. By using relation (2) and Theorem B, we get

$$\|f\|_{B^\Omega_{2,1}} \asymp \Omega^{-1}(2^{-l})\|\delta_l(f,\cdot)\|_2 \asymp \Omega^{-1}(2^{-l})2^{-ld/2}\|\delta_l f^j\|_{l_2^{2^{ld}}}. \tag{13}$$

If the function $f \in L_2 \cap \mathcal{T}_l$ satisfies the relation

$$\|\delta_l f^j\|_{l_2^{2^{ld}}} \ll \Omega(2^{-l})2^{ld/2}, \tag{14}$$

then $C_6 f \in B^\Omega_{2,1} \cap \mathcal{T}_l,\ C_6 > 0.$

In other words, the ball $C_6\Omega(2^{-l})2^{ld/2}B_2^{2^{ld}}$ of radius $C_6\Omega(2^{-l})2^{ld/2}$ is associated with the unit ball from the space $B^\Omega_{2,1} \cap \mathcal{T}_l$. In addition, if $g \in L_q \cap \mathcal{T}_l$, then, by virtue of the Littlewood–Paley theorem and Theorem B, we find

$$\|g\|_q \asymp \|\delta_l(g,\cdot)\|_q \asymp 2^{-ld/q}\|\delta_l g^j\|_{l_q^{2^{ld}}}. \tag{15}$$

In view of relations (12)–(15), we obtain

$$\lambda_m(B^\Omega_{2,1}, L_q) \gg \Omega(2^{-l})2^{ld(1/2-1/q)}\lambda_m\Big(B_2^{2^{ld}}, l_q^{2^{ld}}\Big).$$

It follows from the well-known relation (see [23, p. 209])

$$\lambda_m(B_2^{2^{ld}}, l_q^{2^{ld}}) = d_m(B_{q'}^{2^{ld}}, l_2^{2^{ld}})$$

that

$$\lambda_m(B^\Omega_{2,1}, L_q) \gg \Omega(2^{-l})2^{ld(1/2-1/q)}d_m\Big(B_{q'}^{2^{ld}}, l_2^{2^{ld}}\Big). \tag{16}$$

Further, we need the following auxiliary statement:

Lemma C [24]. *Let $m < n$ and $1 \leq p \leq 2 \leq q < \infty$. Then*

$$d_m(B_p^n, l_q^n) \asymp \max\Big\{n^{1/q-1/p}, \min\big\{1, n^{1/q}m^{-1/2}\big\}\sqrt{1-m/n}\Big\}. \tag{17}$$

Under the imposed conditions, it follows from relation (17) that

$$d_m\left(B_{q'}^{2^{ld}}, l_2^{2^{ld}}\right) \asymp \max\left\{2^{ld(1/2-1/q')}, \min\left\{1, 2^{ld/2}m^{-1/2}\right\}\sqrt{1-\frac{m}{2^{ld}}}\right\}$$

$$\geq \min\left\{1, 2^{ld/2}m^{-1/2}\right\}\sqrt{1-\frac{m}{2^{ld}}}$$

$$\gg \min\left\{1, 2^{ld/2}2^{-(ld-1)/2}\right\}\sqrt{1-\frac{2^{ld}/2}{2^{ld}}} = C_7 > 0. \tag{18}$$

Relations now imply (16) and (18) that

$$\lambda_m\left(B_{2,1}^{\Omega}, L_q\right) \gg \Omega(2^{-l})2^{ld(1/2-1/q)} \asymp \Omega(m^{-1/d})m^{1/2-1/q}.$$

The lower bound is established and, hence, the theorem is proved.

Theorem 3. *Let* $2 \leq p < q < \infty,\ 1 \leq \theta \leq \infty,$ *and let* $\Omega \in \Phi_{\alpha,l},\ \alpha > d(1/p-1/q)$. *Then*

$$\lambda_m\left(B_{p,\theta}^{\Omega}, L_q\right) \asymp \Omega\big(m^{-1/d}\big)m^{1/p-1/q}. \tag{19}$$

Proof. The upper bound of the quantity $\lambda_m\left(B_{p,\theta}^{\Omega}, L_q\right)$ follows from the corresponding estimate for the approximation of functions from the classes $B_{p,\theta}^{\Omega}$ by their cubic Fourier sums [25].

We now establish the lower bound. According to the “inequality of different metrics” (see Theorem D), for $f \in B_{p,\theta}^{\Omega}$ and $p \geq 2$, we get

$$\|f\|_{B_{p,\theta}^{\Omega}} \asymp \left(\sum_{s\in\mathbb{Z}_+}\Omega^{-\theta}(2^{-s})\big\|\sigma_s(f,\cdot)\big\|_p^{\theta}\right)^{1/\theta}$$

$$\ll \left(\sum_{s\in\mathbb{Z}_+}\Omega^{-\theta}(2^{-s})2^{sd\theta(1/2-1/p)}\big\|\sigma_s(f,\cdot)\big\|_2^{\theta}\right)^{1/\theta}$$

$$= \left(\sum_{s\in\mathbb{Z}_+}\Omega_1^{-\theta}(2^{-s})\big\|\sigma_s(f,\cdot)\big\|_2^{\theta}\right)^{1/\theta} \asymp \|f\|_{B_{2,\theta}^{\Omega_1}},$$

where $\Omega_1(\tau) = \Omega(\tau)\tau^{d(1/2-1/p)}$.

It is clear that $\Omega_1 \in \Phi_{\alpha_1,l+1}$,

$$\alpha_1 = \alpha + d(1/2-1/p) > d(1/2-1/q).$$

Thus, for $2 \leq p < \infty$, the embedding $B_{2,\theta}^{\Omega_1} \subset B_{p,\theta}^{\Omega}$ holds. By using this result and Theorem 2, we find

$$\lambda_m(B_{p,\theta}^{\Omega}, L_q) \gg \lambda_m(B_{2,\theta}^{\Omega_1}, L_q) \asymp \Omega_1(m^{-1/d})m^{1/2-1/q} = \Omega(m^{-1/d})m^{1/p-1/q}.$$

Thus, the theorem is proved.

Remark 2. In the one-dimensional case, for the same relationships between the parameters p and q and $2 \le \theta \le q,$ Theorems 2 and 3 were proved in [26] and, for the other values of $\theta,$ in [22].

We now formulate two more statements obtained as corollaries of the well-known results.

Theorem 4. *Let* $1 \le p < q \le 2,\ 1 \le \theta \le \infty,$ *and let* $\Omega \in \Phi_{\alpha,l},\ \alpha > d(1/p - 1/q).$ *Then*

$$\lambda_m\Big(B^{\Omega}_{p,\theta}, L_q\Big) \asymp \Omega\big(m^{-1/d}\big)m^{1/p-1/q}. \tag{20}$$

Theorem 5. *Let* $2 \le q \le p \le \infty,\ (p,q) \ne (\infty,\infty),\ 1 \le \theta \le \infty,$ *and let* $\Omega \in \Phi_{\alpha,l},\ \alpha > 0.$ *Then*

$$\lambda_m\Big(B^{\Omega}_{p,\theta}, L_q\Big) \asymp \Omega\big(m^{-1/d}\big). \tag{21}$$

The upper bounds in (20) and (21) follow from the corresponding estimates for the approximation of functions from the classes $B^{\Omega}_{p,\theta}$ by their cubic Fourier sums [25]. To obtain the lower bounds, we use inequality (3) and the corresponding estimates for the Kolmogorov widths $d_m\Big(B^{\Omega}_{p,\theta}, L_q\Big)$ [20].

Remark 3. For $d = 1,\ 1 < p < q \le 2,$ and $1 \le \theta \le \infty,$ Theorem 4 was proved in [21]. For $p = 1,$ $1 < q \le 2,$ and $1 \le \theta \le q,$ it was proved in [22].

Remark 4. In the one-dimensional case, for $2 \le q \le p < \infty,\ 1 \le \theta \le \infty,$ and $p = \infty,\ 2 \le q < \infty,$ $1 \le \theta \le \infty,$ Theorem 5 was proved in [26] and [22], respectively.

Now let $\Omega(t) = t^r$. By virtue of Theorems 1–5, we arrive at the following statement:

Theorem 6. *Let* $1 \le \theta \le \infty.$ *Then, for* $r > r(d,p,q),$

$$\lambda_m(B^r_{p,\theta}, L_q) \asymp \begin{cases} m^{-r/d}, & 2 \le q \le p \le \infty, \quad (p,q) \ne (\infty,\infty), \\ m^{-r/d+1/p-1/q}, & 1 \le p < q \le 2, \quad 2 \le p < q < \infty, \\ m^{-r/d+1/p-1/2}, & 1 \le p < 2 \le q < p', \\ m^{-r/d+1/2-1/q}, & 1 < p \le 2, \quad p' < q < \infty, \end{cases}$$

where

$$r(d,p,q) = \begin{cases} d(1/p - 1/q)_+, & 2 \le q \le p \le \infty, \quad 1 \le p < q \le 2, \\ & 2 \le p < q < \infty, \\ \max\{d/p, d(1-1/q)\}, & 1 \le p < 2 \le q < p', \\ & 1 < p \le 2, \quad p' < q < \infty, \end{cases}$$

and

$$a_+ = \max\{a, 0\}, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Remark 5. In the one-dimensional case, Theorem 6 was proved by Romanyuk [8–10] (except the cases $p \leq 2,\ 1/p + 1/q < 1$, and $2 \leq p < q < \infty$ for $\theta = \infty$). These case were considered by Galeev [7].

Finally, we make the following conclusions:

Comparing the results obtained in the present paper with the estimates of Kolmogorov widths $d_m(B^{\Omega}_{p,\theta}, L_q)$, we see that the following relation is true under the conditions imposed on the parameters p and q appearing in Theorems 1, 4, and 5:

$$\lambda_m\left(B^{\Omega}_{p,\theta}, L_q\right) \asymp d_m\left(B^{\Omega}_{p,\theta}, L_q\right).$$

At the same timed, if $1 < p \leq 2,\ p' < q < \infty$, and $2 \leq p < q < \infty$, then we get the following order equalities:

$$\lambda_m\left(B^{\Omega}_{p,\theta}, L_q\right) \asymp m^{1-1/q-1/p} d_m\left(B^{\Omega}_{p,\theta}, L_q\right),$$

$$\lambda_m\left(B^{\Omega}_{p,\theta}, L_q\right) \asymp m^{1/p-1/q} d_m\left(B^{\Omega}_{p,\theta}, L_q\right).$$

In [2], the exact-order estimates were established for the linear widths $\lambda_m(B^{\Omega}_{p,\theta}, L_q)$ appearing in Theorems 1–5. However, these estimates were deduced for a narrower spectrum of the smooth parameter α (and, in some cases, for a different spectrum). In addition, the methods used in the present paper to obtain the estimates for the linear widths of the classes $B^{\Omega}_{p,\theta}$ differ from the methods used in [2].

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