Uniform spherical grids via equal area projection from the cube to the sphere

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Abstract

We construct an area preserving map from the cube to the unit sphere $\mathbb{S}^2$, both centered at the origin. More precisely, each face F_i of the cube is first projected to a curved square $\mathcal{S}_i$ of the same area, and then each $\mathcal{S}_i$ is projected onto the sphere by inverse Lambert azimuthal equal area projection, with respect to the points situated at the intersection of the coordinate axes with $\mathbb{S}^2$. This map is then used to construct uniform and refinable grids on a sphere, starting from any grid on a square.

Key words: equal area projection, uniform spherical grid, refinable grid, hierarchical grid.

1 Introduction

The generation of suitable finite subsets of the unit sphere is closely related to all kinds of data approximation of the sphere. Generic approximation problems on the unit sphere include the question of how to find the best element from a defined function set that interpolates or approximates given function values on points of the sphere. Here the question that arises, is how the given function values should be distributed on the sphere in order to achieve optimal approximation results.

In many applications, especially in geosciences and astronomy, but also in medical imaging and computer vision, one requires simple, uniform and refinable grids on the sphere. One simple method to construct such grids is to transfer existing planar grids. A complete description of all known spherical projections from a sphere or parts of a sphere to the plane, used in cartography, is realized in [2, 8].

However, most grid constructions given so far do not provide an equal area partition. But this property is crucial for a series of applications including statistical computations and wavelet constructions, since non-equal area partitions can generate severe distortions at large distances.

The existence of partitions of $\mathbb{S}^2$ into regions of equal area and small diameter has been already used by Alexander [1], who derives lower bounds for the maximum sum of distances between points on the sphere. Based on the construction by Zhou [11], Leopardi derives a recursive zonal equal area sphere partitioning algorithm for the unit sphere $\mathbb{S}^d$ embedded in $\mathbb{R}^{n+1}$, see [4]. The constructed partition in [4] for $\mathbb{S}^2$ consists of polar cups and rectilinear regions that are arranged in zonal collars. Besides the problem that we have to deal with different kinds of areas, the obtained partition is not suitable for various applications where one needs to avoid that vertices of spherical rectangles lie on edges of neighboring rectangles.

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Other constructions for equal area partitions of $\mathbb{S}^2$ used in astronomy include the HEALPix grid [3], providing a hierarchical equal area iso-latitude pixelation, and the icosahedron-based method by Tegmark [10], see also [9]. In [7], an equal area global partition method based upon circle edges is presented. Starting with a spherical triangulation, obtained e.g. by projecting the faces of an icosahedron to the sphere, a subdivision method is proposed to partition each spherical triangle into four equal area subtriangles.

In [5], one of the authors suggested a new area preserving projection method based on a mapping of the square onto a disc in a first step, followed by a lifting to the sphere by the inverse Lambert projection. This idea can also be generalized to construct uniform and refinable grids on elliptic domains and on some surfaces of revolution, see [6].

In this paper we construct an area preserving map from a cube to the unit sphere $\mathbb{S}^2$. Thus, *any* grid on the cube can be transported to the sphere. The new construction admits to transform well-known techniques from the square to (parts of) the sphere. For example, the well-known tensor-product wavelet transform for data analysis and denoising can be simply transferred to the sphere, where distortions are negligible because of the equal area property. Further, since arbitrary grids on the cube resp. on the square can now simply be transported to the sphere, we believe that this construction may achieve an essential impact for different applications in geosciences. Since we give explicit formulas both for the map from the cube to the sphere, and from the sphere to the cube, the method is easy to implement.

Our new construction scheme consists of two steps. In the first step, we construct in Section 3 a bijection T from each face F_i of the cube, onto a curved square $\mathcal{S}_i$. In Section 4 we construct the inverse T^{-1}. In the second step, we combine T with the inverse Lambert azimuthal projection, in order to map each face F_i of the cube onto a subset $\mathcal{F}_i$ of the sphere, such that $\cup_{i=1}^{6}\mathcal{F}_i = \mathbb{S}^2$. Finally, we present some examples of the obtained spherical grids.

2 Preliminaries

Consider the unit sphere $\mathbb{S}^2$ centered at the origin O and the cube $\mathbb{K}$ centered at O, with the same area. Thus, the edge of the cube has the length $a = \sqrt{2\pi/3}$. Denote $\beta = a/2 = \sqrt{\pi/6}$.

We cut the sphere with the six diagonal planes $z = \pm x$, $z = \pm y$, $y = \pm x$ and obtain the curves in Figure 1.

Let us focus on one of these curves $\mathcal{C}_1$, given by the equations

$$\begin{cases} x^2 + y^2 + z^2 = 1, \\ z = x > 0, \\ y \in \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right]. \end{cases}$$

Using the parameter $t \in [-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$, the curve $\mathcal{C}_1$ is described by

$$(x, y, z) = (\sqrt{\frac{1}{2}(1-t^2)},\, t,\, \sqrt{\frac{1}{2}(1-t^2)}).$$

The Lambert azimuthal equal-area projection of the sphere $x^2 + y^2 + z^2 = 1$ onto the plane $z = 1$ is given by

$$(x_L, y_L) = \left(\sqrt{\frac{2}{1+z}}\, x, \sqrt{\frac{2}{1+z}}\, y \right). \tag{1}$$

Hence, the Lambert projection of the curve $\mathcal{C}_1$ onto the plane $z = 1$ has the equations

$$x_L = \frac{\sqrt{2-2t^2}}{\sqrt{2+\sqrt{2-2t^2}}}, \tag{2}$$

$$y_L = \frac{2t}{\sqrt{2+\sqrt{2-2t^2}}}, \tag{3}$$

$$z_L = 1,$$

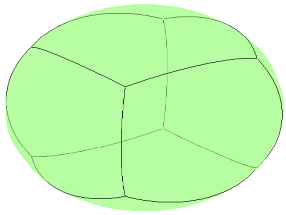

Figure 1: The curves of intersection of the cube with the diagonal plans.

with $t \in [-1/\sqrt{3}, 1/\sqrt{3}]$. The calculations show that

$$x_L^2 + y_L^2 = 2 - \sqrt{2 - 2t^2}, \quad (4)$$

$$\frac{y_L}{x_L} = \frac{\sqrt{2}t}{\sqrt{1-t^2}}. \quad (5)$$

We denote by $\mathcal{S}_a$ the curved square in the tangent plane $z = 1$, formed by the curved edge of equations (2)-(3) and three other curved edges obtained in the same manner by intersecting the sphere with the half-planes $z = -x > 0$, $z = y > 0$ and $z = -y > 0$, respectively (Figure 2). Because of the area-preserving property of the Lambert projection, the area $\mathcal{A}(\mathcal{S}_a)$ of the obtained curved square $\mathcal{S}_a$ is exactly $\frac{2\pi}{3} = a^2$. With the help of this projection, we have simplified the problem of finding an area preserving map from the cube to the sphere $\mathbb{S}^2$ to a problem of finding a two-dimensional map from a square, i.e., a face of the cube, to the curved square $\mathcal{S}_a$.

3 Mapping a square onto a curved square

In this section we derive an area preserving bijection $T : \mathbb{R}^2 \to \mathbb{R}^2$, which maps the square

$$S_a = \{(x, y) \in \mathbb{R}^2, |x| \leq a,\ |y| \leq a\}$$

onto the curved square $\mathcal{S}_a$ that has been constructed in Section 2. Here, we say that T is area-preserving, if it has the property

$$\mathcal{A}(D) = \mathcal{A}(T(D)),\ \text{for every domain } D \subseteq \mathbb{R}^2. \quad (6)$$

Since we consider only the two-dimensional problem in this section, we will work in the (x, y)-plane for simplicity, i.e., the curved boundary $\mathcal{C}_a$ of $\mathcal{S}_a$ is given by (2) and (3) with the notation $x = x_L$ and $y = y_L$, and analogously for the three other curves.

We focus for the moment on the first octant of the (x, y)-plane

$$I = \{(x, y) \in \mathbb{R}^2,\ 0 < y \leq x\}$$

with origin $O = (0, 0)$ and take a point $M = (x_M, y_M) = (x_M, mx_M) \in I$, where m is a parameter with $0 \leq m \leq 1$, see Figure 2. The map T will be defined in such a way that each half-line d_m of equation $y = m\,x$ $(0 \leq m \leq 1)$ is mapped onto the half-line $d_{\varphi(m)}$ of equation $y = \varphi(m)x$, where $\varphi : [0, 1] \to \mathbb{R}$ is a differentiable function that satisfies

$$\varphi(0) = 0,\ \varphi(1) = 1 \text{ and } 0 \leq \varphi(m) \leq 1. \quad (7)$$

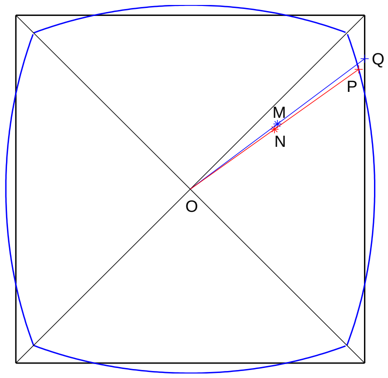

Figure 2: The square S_a, the curved square $\mathcal{S}_a$ and the action of the transform T. Points N and P are the images of M and Q.

This condition ensures that the diagonal of the square S_a and the axis x are invariant under φ. We denote by (x_N, y_N) the coordinates of the point $N = T(M)$. Let Q be the intersection of OM with the square S_a (see Figure 2). The point Q has the coordinates $(x_Q, y_Q) = (\beta, m\beta)$, where $\beta = \frac{a}{2} = \sqrt{\frac{\pi}{6}}$, and the line ON has the equation $y = \varphi(m)x$. Further, let the point $P = (x_P, y_P) = P(x_P, \varphi(m)x_P)$ be the intersection of ON with the curved square $\mathcal{S}_a$. Thus, the coordinates of P satisfy the equations (4) and (5) with some $t_P \in [0, \frac{1}{\sqrt{3}}]$, and from (5) we have

$$\varphi(m) = \frac{\sqrt{2}t_P}{\sqrt{1-t_P^2}}, \text{ whence } t_P = \frac{\varphi(m)}{\sqrt{2+\varphi^2(m)}}.$$

Replacing t_P in (2) and (3), we obtain the coordinates of P in the form

$$\begin{aligned} x_P &= \frac{\sqrt{2}}{\sqrt{2+\varphi^2(m)}\sqrt{1+\frac{1}{\sqrt{2+\varphi^2(m)}}}}, \\ y_P &= \frac{\sqrt{2}\,\varphi(m)}{\sqrt{2+\varphi^2(m)}\sqrt{1+\frac{1}{\sqrt{2+\varphi^2(m)}}}}. \end{aligned}$$

Some simple calculations yield the distances

$$\begin{aligned} OM &= x_M\sqrt{1+m^2}, \\ OQ &= \beta\sqrt{1+m^2}, \\ ON &= x_N\sqrt{1+\varphi^2(m)}, \\ OP &= \left(2 - \frac{2}{\sqrt{2+\varphi^2(m)}}\right)^{1/2}. \end{aligned}$$

We want to determine the map T such that

$$\frac{ON}{OP} = \frac{OM}{OQ}.$$

From the above calculations we then obtain

$$
\begin{aligned}
x_N &= \frac{x}{\beta}\frac{1}{\sqrt{1+\varphi^2(m)}}\sqrt{2-\frac{2}{\sqrt{2+\varphi^2(m)}}},\\
y_N &= \varphi(m)x_N = \frac{x}{\beta}\frac{\varphi(m)}{\sqrt{1+\varphi^2(m)}}\sqrt{2-\frac{2}{\sqrt{2+\varphi^2(m)}}}.
\end{aligned}
$$

With this assertion, the map T is now completely described by means of the function φ, and we obtain that T maps the point $(x,y)\in I$ onto the point (X,Y) given by

$$
X = \frac{x}{\beta}\frac{1}{\sqrt{1+\varphi^2(\frac{y}{x})}}\sqrt{2-\frac{2}{\sqrt{2+\varphi^2(\frac{y}{x})}}}, \tag{8}
$$

$$
Y = \frac{x}{\beta}\frac{\varphi(\frac{y}{x})}{\sqrt{1+\varphi^2(\frac{y}{x})}}\sqrt{2-\frac{2}{\sqrt{2+\varphi^2(\frac{y}{x})}}}. \tag{9}
$$

Next we impose the area preserving property by a suitable determination of φ. For this purpose, we define the function φ such that the Jacobian of T is 1. After simplification, the Jacobian writes as

$$
J(T) = \det\begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y}\end{pmatrix} = \frac{2}{\beta^2}\,\frac{\varphi'\left(\frac{y}{x}\right)}{2+\varphi^2(\frac{y}{x})+\sqrt{2+\varphi^2\left(\frac{y}{x}\right)}}.
$$

For solving the equation $J(T)=1$ we substitute $v := \frac{y}{x}$, and thus, in the considered case $0<y\le x$ we have $v\in(0,1]$. Hence, with the simplified notation $\varphi=\varphi(v)$, we get

$$
\frac{\varphi'}{2+\varphi^2+\sqrt{2+\varphi^2}} = \frac{\beta^2}{2}.
$$

Integration gives

$$
\arctan\varphi - \arctan\frac{\varphi}{\sqrt{2+\varphi^2}} = \frac{\beta^2}{2}v + C.
$$

The condition $\varphi(0)=0$ yields $C=0$. Next, in order to determine φ we use the formula

$$
\arctan a - \arctan b = \arctan\frac{a-b}{1+ab} \qquad \forall a,b\in\mathbb{R},\ ab>-1
$$

and we further obtain

$$
\frac{\varphi\,(\sqrt{2+\varphi^2}-1)}{\sqrt{2+\varphi^2}+\varphi^2} = \tan\frac{\beta^2 v}{2}. \tag{10}
$$

To simplify this term, we introduce the notation $\gamma = \tan\frac{\beta^2 v}{2} = \tan\frac{\pi}{12}v$, and equality (10) yields

$$
\sqrt{2+\varphi^2}\,(\varphi-\gamma) = \varphi(\gamma\varphi+1). \tag{11}
$$

From the requirements (7) we can deduce that, for $(x,y)\in I$ with $y>0$, both sides of equality (11) are positive. Thus, (11) is equivalent with

$$
\begin{gathered}
(2+\varphi^2)(\varphi-\gamma)^2 = \varphi^2(\gamma\varphi+1)^2\\
(\varphi^2+1)((1-\gamma^2)\varphi^2-4\varphi\gamma+2\gamma^2)=0,
\end{gathered}
$$

which gives

$$
\varphi_{1,2}(v) = \frac{2\gamma\pm\sqrt{2}\gamma\sqrt{1+\gamma^2}}{1-\gamma^2}.
$$

We denote $\alpha = \frac{\beta^2}{2} v = \frac{\beta^2 y}{2x}$. Thus $\gamma = \tan\alpha$ and some calculations show that the function φ becomes

$$\varphi_{1,2}(v) = \varphi_{1,2}\left(\frac{y}{x}\right) = \frac{\sqrt{2}\sin\alpha}{\sqrt{2}\cos\alpha \pm 1}, \qquad \text{with } \alpha = \frac{\pi y}{12x}.$$

If we take into account the condition $\varphi(1) = 1$ and the equality

$$\sqrt{2}\sin\frac{\pi}{12} = \sqrt{2}\cos\frac{\pi}{12} - 1,$$

the only convenient solution is

$$\varphi\left(\frac{y}{x}\right) = \frac{\sqrt{2}\sin\alpha}{\sqrt{2}\cos\alpha - 1}, \qquad \text{with } \alpha = \frac{\pi y}{12x}.$$

Finally, we need to replace φ in the equations (8)-(9). Some straightforward calculations show that

$$\begin{aligned}
\sqrt{\varphi^2+2} &= \frac{2-\sqrt{2}\cos\alpha}{\sqrt{2}\cos\alpha - 1},\\
\sqrt{\varphi^2+1} &= \frac{\sqrt{3-2\sqrt{2}\cos\alpha}}{\sqrt{2}\cos\alpha - 1},\\
\frac{\varphi}{\sqrt{\varphi^2+1}} &= \frac{\sqrt{2}\sin\alpha}{\sqrt{3-2\sqrt{2}\cos\alpha}}.
\end{aligned}$$

Hence, for $(x,y) \in I$, the formulas for the desired area preserving map $T(x,y) = (X,Y)$ are given by

$$\begin{aligned}
X &= 2^{1/4}\frac{x}{\beta}\,\frac{\sqrt{2}\cos\alpha - 1}{\sqrt{\sqrt{2}-\cos\alpha}},\\
Y &= 2^{1/4}\frac{x}{\beta}\,\frac{\sqrt{2}\sin\alpha}{\sqrt{\sqrt{2}-\cos\alpha}}, \text{ with } \alpha = \frac{y\pi}{12x}, \quad \beta = \sqrt{\frac{\pi}{6}}.
\end{aligned}$$

Similar arguments for the other seven octants show that the area preserving map $T : \mathbb{R}^2 \to \mathbb{R}^2$ which maps squares into curved squares is defined as follows:

For $|y| \le |x|$,

$$(x,y) \longmapsto (X,Y) = \left(\frac{2^{1/4}x}{\beta}\,\frac{\sqrt{2}\cos\frac{y\pi}{12x} - 1}{\sqrt{\sqrt{2}-\cos\frac{y\pi}{12x}}}, \frac{2^{1/4}x}{\beta}\,\frac{\sqrt{2}\sin\frac{y\pi}{12x}}{\sqrt{\sqrt{2}-\cos\frac{y\pi}{12x}}}\right); \tag{12}$$

For $|x| \le |y|$,

$$(x,y) \longmapsto (X,Y) = \left(\frac{2^{1/4}y}{\beta}\,\frac{\sqrt{2}\sin\frac{x\pi}{12y}}{\sqrt{\sqrt{2}-\cos\frac{x\pi}{12y}}}, \frac{2^{1/4}y}{\beta}\,\frac{\sqrt{2}\cos\frac{x\pi}{12y} - 1}{\sqrt{\sqrt{2}-\cos\frac{x\pi}{12y}}}\right). \tag{13}$$

Figure 3 shows two grids on the curved square.

4 The inverse map

To make the area preserving map T applicable in practice, we need also to derive a closed simple form for the inverse mapping T^{-1}. Let us consider first the case when $x \ge y > 0$, when the map T is given by formula (12).

With the notation

$$\eta = \cos\frac{y\pi}{12x}, \tag{14}$$

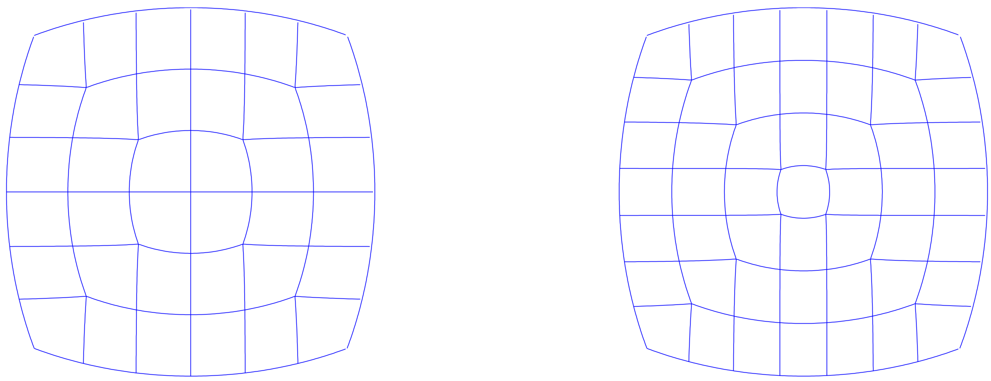

Figure 3: Two grids on the curved square, with seven and eight equidistant horizontal and vertical lines.

we have

$$X^2 = \frac{\sqrt{2}x^2}{\beta^2}\frac{(\sqrt{2}\eta-1)^2}{\sqrt{2}-\eta}, \qquad Y^2 = \frac{\sqrt{2}x^2}{\beta^2}\frac{2(1-\eta^2)}{\sqrt{2}-\eta}.$$

In particular,

$$\begin{aligned} X^2+Y^2 &= \frac{\sqrt{2}x^2}{\beta^2}\frac{3-2\sqrt{2}\eta}{\sqrt{2}-\eta}, \\ X^2+\frac{1}{2}Y^2 &= \frac{\sqrt{2}x^2}{\beta^2}(\sqrt{2}-\eta), \\ X^2-\frac{1}{2}Y^2 &= \frac{\sqrt{2}x^2}{\beta^2}\frac{\eta(3\eta-2\sqrt{2})}{\sqrt{2}-\eta}. \end{aligned} \tag{15}$$

We define

$$B := \frac{X^2+\frac{1}{2}Y^2}{X^2+Y^2} = \frac{(\sqrt{2}-\eta)^2}{3-2\sqrt{2}\eta}.$$

Hence, we obtain the equation in η

$$\eta^2 - 2\sqrt{2}(1-B)\eta + 2 - 3B = 0$$

with the solutions

$$\eta_{1,2} = \sqrt{2}(1-B) \pm \sqrt{B(2B-1)}.$$

In our case, $\eta \in (\cos\frac{\pi}{12}, 1)$, therefore the only convenient solution is $\eta = \sqrt{2}(1-B) + \sqrt{B(2B-1)}$. Then, from

$$1-B = \frac{Y^2}{2(X^2+Y^2)} \quad \text{and} \quad 2B-1 = \frac{X^2}{X^2+Y^2}$$

it follows that

$$\eta = \frac{Y^2 + X\sqrt{2X^2+Y^2}}{\sqrt{2}(X^2+Y^2)}. \tag{16}$$

For simplicity we introduce the notation $w := Y/X$. Thus, we have $w \in [0,1]$ and

$$\frac{\pi}{12}\frac{y}{x} = \arccos\eta = \arccos\left(\frac{w^2}{\sqrt{2}(1+w^2)} + \frac{\sqrt{2+w^2}}{\sqrt{2}(1+w^2)}\right).$$

Now we use the identity

$$\arccos(ab + \sqrt{1-a^2}\sqrt{1-b^2}\,) = \arccos a - \arccos b, \quad \forall \quad 0 \le a \le b \le 1,$$

for $a = \frac{w}{\sqrt{2(1+w^2)}}$ and $b = \frac{w}{\sqrt{1+w^2}}$ and obtain

$$\frac{\pi}{12}\frac{y}{x} = \arccos \frac{w}{\sqrt{2(1+w^2)}} - \arccos \frac{w}{\sqrt{1+w^2}}.$$

Other possible expressions are

$$\begin{aligned} \frac{\pi}{12}\frac{y}{x} &= \arctan \frac{\sqrt{2+w^2}}{w} - \arctan \frac{1}{w} \\ &= \arctan w - \arctan \frac{w}{\sqrt{2+w^2}} \\ &= \arctan \frac{Y}{X} - \arctan \frac{Y}{\sqrt{2X^2+Y^2}}. \end{aligned} \tag{17}$$

For the calculation of x from X and Y, we use the second equality in (15) and $\eta = \frac{w^2+\sqrt{2+w^2}}{\sqrt{2}(1+w^2)}$ in order to find

$$\begin{aligned} x^2 &= \frac{\beta^2}{\sqrt{2}}\,\frac{X^2+\frac{Y^2}{2}}{\sqrt{2}-\eta} = \frac{\beta^2 X^2}{2\sqrt{2}}\,\frac{2+w^2}{\sqrt{2}-\eta} \\ &= \frac{\beta^2 X^2}{2}\,\frac{(1+w^2)\sqrt{2+w^2}}{\sqrt{w^2+2}-1} \\ &= \frac{\beta^2 X^2}{2}\sqrt{2+w^2}\,(\sqrt{2+w^2}+1) \\ &= \frac{\beta^2}{2}\sqrt{2X^2+Y^2}\,(\sqrt{2X^2+Y^2}+X). \end{aligned}$$

Finally, from (14) and (17) we find

$$\begin{aligned} y &= \frac{12x}{\pi}\arccos\eta = \frac{12x}{\pi}\left(\arctan w - \arctan \frac{w}{\sqrt{2+w^2}}\right) \\ &= \frac{12}{\pi}\frac{\beta X}{\sqrt{2}}\sqrt[4]{2+w^2}\sqrt{1+\sqrt{2+w^2}}\left(\arctan w - \arctan \frac{w}{\sqrt{2+w^2}}\right) \\ &= \frac{\sqrt{2}}{\beta}\sqrt[4]{2X^2+Y^2}\sqrt{X+\sqrt{2X^2+Y^2}}\left(\arctan \frac{Y}{X} - \arctan \frac{Y}{\sqrt{2X^2+Y^2}}\right). \end{aligned}$$

Using similar arguments for the other seven octants, we obtain that the inverse $T^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ has the expressions

For $|Y| \le |X|$,

$$\begin{aligned} (X,Y) \longmapsto (x,y) = \Bigg(&\tfrac{\beta}{\sqrt{2}}\operatorname{sign}(X)\sqrt[4]{2X^2+Y^2}\sqrt{|X|+\sqrt{2X^2+Y^2}}, \\ &\tfrac{\sqrt{2}}{\beta}\sqrt[4]{2X^2+Y^2}\sqrt{|X|+\sqrt{2X^2+Y^2}}\Big(\operatorname{sign}(X)\arctan\tfrac{Y}{X} - \arctan\tfrac{Y}{\sqrt{2X^2+Y^2}}\Big)\Bigg); \end{aligned}$$

For $|X| \le |Y|$,

$$\begin{aligned} (X,Y) \longmapsto (x,y) = \Bigg(&\tfrac{\sqrt{2}}{\beta}\sqrt[4]{X^2+2Y^2}\sqrt{|Y|+\sqrt{X^2+2Y^2}}\Big(\operatorname{sign}(Y)\arctan\tfrac{X}{Y} - \arctan\tfrac{X}{\sqrt{X^2+2Y^2}}\Big), \\ &\tfrac{\beta}{\sqrt{2}}\operatorname{sign}(Y)\sqrt[4]{X^2+2Y^2}\sqrt{|Y|+\sqrt{X^2+2Y^2}}\Bigg). \end{aligned}$$

5 Mapping the curved squares onto the sphere

The complete mapping from the cube to the sphere $\mathbb{S}^2$ is now described in two steps. In the first step, each face F_i of the cube $\mathbb{K}$ will be mapped onto a domain $\widetilde{F_i}$, bounded by a curved square, using the transformation T. In the second step, each $\widetilde{F_i}$ will be mapped onto $\mathcal{F}_i \subseteq \mathbb{S}^2$ by the inverse Lambert azimuthal projection, with respect to the center of F_i. Obviously

$$\bigcap_{i=1}^{6} \mathcal{F}_i = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{6} \mathcal{F}_i = \mathbb{S}^2.$$

We denote by $L_{(0,0,1)}$ the Lambert azimuthal area preserving projection with respect to the North Pole $N = (0,0,1)$. Remember that $L_{(0,0,1)}$ maps a point $(x,y,z) \in \mathbb{S}^2$ onto the point $(X_L, Y_L, 1)$ situated in the tangent plane at N, as given in (1).

Applying the inverse Lambert projection $L^{-1}_{(0,0,1)}$, the point $(X,Y,1)$, situated in the tangent plane to $\mathbb{S}^2$ at the pole N, maps onto $(x_L, y_L, z_L) \in \mathbb{S}^2$ given by

$$x_L = \sqrt{1 - \frac{X^2+Y^2}{4}}\, X, \tag{18}$$

$$y_L = \sqrt{1 - \frac{X^2+Y^2}{4}}\, Y, \tag{19}$$

$$z_L = 1 - \frac{X^2+Y^2}{2}. \tag{20}$$

Thus, the application $T \circ L^{-1}_{(0,0,1)}$ maps the upper face of the cube projects onto $\mathcal{F}_1 \subset \mathbb{S}^2$. The formulas for $T \circ L^{-1}_{(0,0,1)}(x,y)$ are given by (18), (19) and (20), where X, Y are calculated with formulas (12), (13). One can easily obtain similar formulas for $T \circ L^{-1}_{(0,0,-1)}$, $T \circ L^{-1}_{(0,1,0)}$, $T \circ L^{-1}_{(0,-1,0)}$, $T \circ L^{-1}_{(1,0,0)}$ and $T \circ L^{-1}_{(-1,0,0)}$. Figure 4 shows some grids of the sphere, where a regular partition of the faces of the cube into 49, 64, 121 and 144 equal area squares has been applied.

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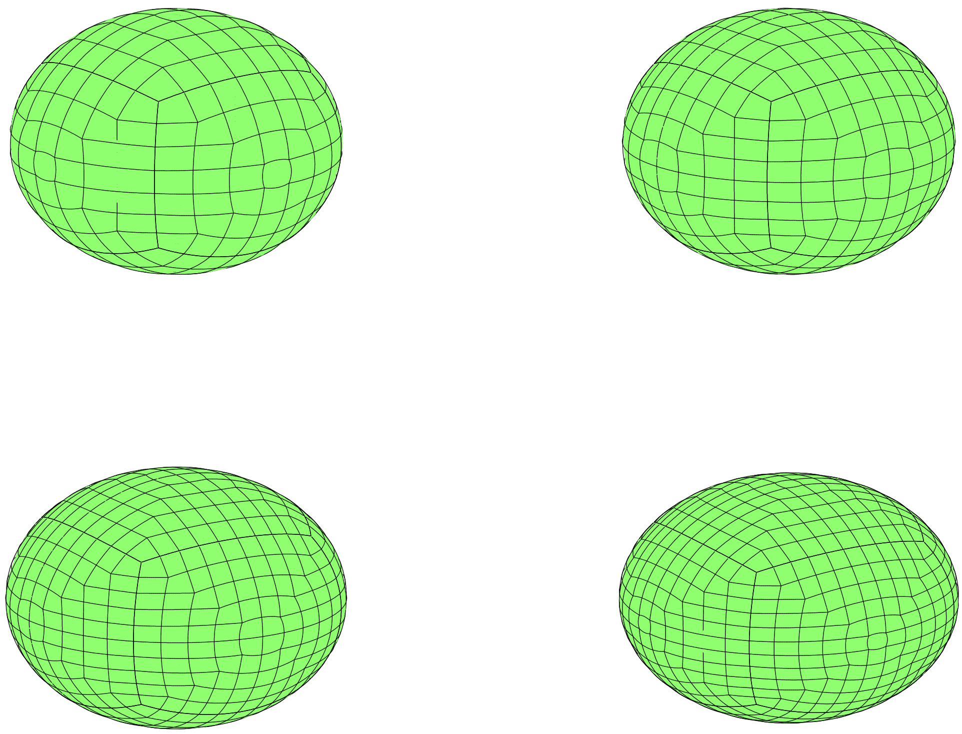

Figure 4: Grids on the sphere.

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