
Non-negativity constraints in the one-dimensional discrete-time phase retrieval problem

Robert Beinert

Institut für Numerische und Angewandte Mathematik
Georg-August-Universität Göttingen

Abstract: Phase retrieval problems occur in a wide range of applications in physics and engineering such as crystallography, astronomy, and laser optics. Common to all of them is the recovery of an unknown signal from the intensity of its Fourier transform. Because of the well-known ambiguousness of these problems, the determination of the original signal is generally challenging. Although there are many approaches in the literature to incorporate the assumption of non-negativity of the solution into numerical algorithms, theoretical considerations about the solvability with this constraint occur rarely. In this paper, we consider the one-dimensional discrete-time setting and investigate whether the usually applied a priori non-negativity can overcome the ambiguousness of the phase retrieval problem or not. We show that the assumed non-negativity of the solution is usually not a sufficient a priori condition to ensure uniqueness in one-dimensional phase retrieval. More precisely, using an appropriate characterization of the occurring ambiguities, we show that neither the uniqueness nor the ambiguousness are rare exceptions.

Key words: Phase retrieval; One-dimensional signals; Compact support; Non-negativity constraints

AMS Subject classifications: 42A05, 94A08, 94A12

1. Introduction

In many fields of physics and engineering, one is faced with the recovery of an unknown signal only from the intensity of its Fourier transform. This phase retrieval problem occurs in different applications as crystallography [Mil90, Hau91], astronomy [BS79, DF87] and laser optics [SSTo4, SSD⁺o6]. In general, the recovery of an analytic or numerical solution is challenging because of the well-known ambiguousness of the problem. Therefore, it is of essential importance to employ suitable a priori information about the original signal in order to find a meaningful solution or, in the best case, the original signal itself.

In the rich literature on the phase retrieval problem, there are different approaches to reduce the solution set or to ensure uniqueness. For instance, the unknown signal x can be superposed with an appropriate reference signal h such that one has access to the additional Fourier intensity of $x + h$. This idea was studied in [KH90a, KH90b, BP15] for a known and in [KH93, RDN13, BP15, Bei16] for an unknown reference signal. More particular reference signals have been considered in [BFGR76, CESV13, Bei16]. Instead

of interference measurements, it is also possible to use additional measurements in the time domain. For instance, one can employ additional magnitudes or phases to ensure a unique recovery of the desired signal [LTo8, BP16].

In the last years, the phase retrieval problem has been generalized from the classical setting to the recovery of an finite-dimensional vector x from appropriate frame measurements $|\langle x, v_k \rangle|$. Here the question arises how the underlying frame vectors have to be chosen, and how many frame vectors are needed to ensure the recovery of x , see for instance [BCEo6, BBCEo9, BCM14, BH15] and references therein.

In this paper, we consider the one-dimensional phase retrieval problem for discrete-time signals, where we restrict ourselves to the recovery of an unknown signal with finite support. Here the occurring ambiguities can be explicitly specified by an appropriate factorization of the autocorrelation signal, see [BS79, BP15]. Additionally, we assume that the unknown signal is real-valued and non-negative. This a priori constraint is usually applied if the unknown signal represents some intensity, see for instance [Fie78, BS79, DF87, SSD⁺o6, LP14] and references therein. Although there are many efforts to incorporate the non-negativity into numerical algorithms, the solvability under this constraint is studied rarely. For this purpose, we consider the issue whether the usually applied a priori non-negativity can overcome the ambiguousness of the phase retrieval problem or not.

The paper is organized as follows. In section 2, we introduce the one-dimensional discrete-time phase retrieval problem and briefly recall the characterization of the occurring ambiguities in [BP15]. Here we distinguish between negligible, trivial ambiguities, like reflection and time shifts, and non-trivial ambiguities. Based on this characterization, we derive appropriate conditions whether a solution is non-negative or not by exploiting that the Fourier transform of a finite-supported signal is mainly an algebraic polynomial, see section 3. Transferring our observation to the complete solution set, we can explicitly construct phase retrieval problems that are uniquely solvable or have a certain number of non-trivial non-negative solutions, see section 4. Finally, in section 5, we present our main result that neither the ambiguousness nor the uniqueness are rare exceptions, and that the non-negativity thus is not sufficient to ensure the unique recovery of the desired signal.

2. The phase retrieval problem

In the following, we consider the *one-dimensional discrete-time phase retrieval problem*. This variant of the phase retrieval problem consists in the recovery of an unknown discrete-time signal $x := (x[n])_{n \in \mathbb{Z}}$ from its Fourier intensity $|\hat{x}|$, where the discrete-time Fourier transform is given by

$$\hat{x}(\omega) := \mathcal{F}[x](\omega) := \sum_{n \in \mathbb{Z}} x[n] e^{-i\omega n} \quad (\omega \in \mathbb{R}).$$

Further, we assume that the unknown signal x has a finite support and that all components $x[n]$ are non-negative.

Similarly to the recovery of a complex-valued signal, the phase retrieval problem for non-negative signals always possesses some negligible ambiguities. More precisely, we can simply transfer [BP15, Proposition 2.1] to non-negative signals.

Proposition 2.1. *Let x be a non-negative signal with finite support. Then*

- (i) *the time shifted signal $(x[n - n_0])_{n \in \mathbb{Z}}$ for $n_0 \in \mathbb{Z}$*
- (ii) *the reflected signal $(x[-n])_{n \in \mathbb{Z}}$*

have the same Fourier intensity $|\widehat{x}|$.

Consequently, the applied assumption that the unknown signal is non-negative cannot ensure uniqueness of the discrete-time phase retrieval problem. However, since the shift and the reflection in Proposition 2.1 are closely related to the original signal, we call these negligible ambiguities *trivial*. Unfortunately, besides this trivial ambiguities, our phase retrieval problem can have further non-trivial ambiguities as exemplarily shown in [BS79, Example 1, et seqq.] and [Fie78, Figure 2]. In order to decide whether these examples are rare exceptions or the general case, we adapt the characterization of the complete solution set in [BP15] to our specific problem.

For this purpose, we recall that the *autocorrelation signal* a of a signal x is given by

$$a[n] := \sum_{k \in \mathbb{Z}} \overline{x[k]} x[k + n] \quad (n \in \mathbb{Z}),$$

and that the squared Fourier intensity can be written as

$$|\widehat{x}(\omega)|^2 = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x[n] \overline{x[k]} e^{-i\omega(n-k)} = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x[k + n] \overline{x[k]} e^{-i\omega n} = \widehat{a}(\omega).$$

Since x has a finite support, this property is transferred to the autocorrelation signal. Furthermore, the definition immediately implies that the components of a have to be symmetric, i.e., $a[-n] = a[n]$ for $n \in \mathbb{Z}$. Thus, the *autocorrelation function* \widehat{a} is here always an even non-negative trigonometric polynomial of degree $N - 1$, where N denotes the support length of the signal x . Since a trigonometric polynomial is completely determined by finitely many samples at appropriate points, it is not necessary to know the Fourier intensity $|\widehat{x}(\omega)|$ for all $\omega \in \mathbb{R}$. Indeed, the complete Fourier intensity of a real signal with support length N is already defined by N samples in the interval $[0, \pi)$.

Following the lines in [BP15], we define the *associated polynomial* P to the trigonometric polynomial \widehat{a} by

$$P(z) := \sum_{n=0}^{2N-2} a[n - N + 1] z^n$$

such that $\widehat{a}(\omega) = e^{i\omega(N-1)} P(e^{-i\omega})$. Since the coefficients of P are real and still satisfy $a[-n] = a[n]$, the zeros of P have a special structure. More precisely, the real zeros occur in pairs $(\gamma, \bar{\gamma}^{-1})$ and the complex zeros in quads $(\gamma, \bar{\gamma}, \gamma^{-1}, \bar{\gamma}^{-1})$ or in the two pairs $(\gamma, \bar{\gamma}^{-1})$ and $(\bar{\gamma}, \gamma^{-1})$. Thus, the associated polynomial can always be written in the form

$$P(z) = a[N-1] \prod_{j=1}^{N-1} (z - \gamma_j) (z - \bar{\gamma}_j^{-1}).$$

Based on this observation, we can factorize the even non-negative polynomial \widehat{a} by

$$\begin{aligned} \widehat{a}(\omega) &= |P(e^{-i\omega})| = |a[N-1]| \prod_{j=1}^{N-1} |e^{-i\omega} - \gamma_j| |e^{-i\omega} - \bar{\gamma}_j^{-1}| \\ &= |a[N-1]| \prod_{j=1}^{N-1} |e^{-i\omega} - \gamma_j| |\gamma_j|^{-1} |\bar{\gamma}_j - e^{i\omega}| \\ &= |a[N-1]| \prod_{j=1}^{N-1} |\gamma_j|^{-1} \cdot \left| \prod_{j=1}^{N-1} (e^{-i\omega} - \gamma_j) \right|^2, \end{aligned}$$

which yields the following characterization of the solution set, see [BP15, Theorem 2.4].

Theorem 2.2. *Let \widehat{a} be an even non-negative trigonometric polynomial of degree $N - 1$. Then, each solution x of the discrete-time phase retrieval problem $|\widehat{x}|^2 = \widehat{a}$ with finite support and non-negative components has a Fourier representation of the form*

$$\widehat{x}(\omega) = e^{-i\omega n_0} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot \prod_{j=1}^{N-1} (e^{-i\omega} - \beta_j), \quad (1)$$

where n_0 is an integer, and where for each j the value β_j is chosen from the zero pair $(\gamma_j, \bar{\gamma}_j^{-1})$ of the associated polynomial to \widehat{a} .

Thus, each solution x of the discrete-time phase retrieval problem $|\widehat{x}|^2 = \widehat{a}$ is uniquely given by the shift parameter n_0 and the chosen values β_j . Since $B := \{\beta_1, \dots, \beta_{N-1}\}$ is a subset of the zero set of the associated polynomial P , we call B the *corresponding zero set* of the solution x . Besides the trivial shift ambiguity, which is directly encoded in (1) by the factor $e^{-i\omega n_0}$, Theorem 2.2 covers the reflection ambiguity too. More precisely, one can show that the reflection $x[-\cdot]$ corresponds to the reflected zero set $\{\bar{\beta}_1^{-1}, \dots, \bar{\beta}_{N-1}^{-1}\}$ if x corresponds to $\{\beta_1, \dots, \beta_{N-1}\}$. Consequently, the discrete-time phase retrieval problem to recover a non-negative signal x with support length N can have at most 2^{N-2} non-trivially different solutions.

3. Algebraic polynomials with non-negative coefficients

To answer the question whether the phase retrieval problem in Theorem 2.2 can have more than one non-negative non-trivial solution, we investigate conditions on the zero set $B := \{\beta_1, \dots, \beta_{N-1}\}$ which ensure that a real signal with finite support possesses only non-negative components. We notice that the signal x in (1) is non-negative if and only if all coefficients of the monic polynomial

$$Q(z) := \prod_{j=1}^{N-1} (z - \beta_j)$$

are non-negative. Using Vieta's formulae and the *elementary symmetric polynomials* S_n defined by

$$S_n(\beta_1, \dots, \beta_{N-1}) := \sum_{1 \leq k_1 < \dots < k_n \leq N-1} \beta_{k_1} \cdots \beta_{k_n} \quad (n = 1, \dots, N-1)$$

as well as $S_0 := 1$ and $S_n := 0$ for $n < 0$ and $n \geq N$, we obtain the representation

$$Q(z) = \sum_{n=0}^{N-1} (-1)^n S_n(\beta_1, \dots, \beta_{N-1}) z^{N-1-n}.$$

The theorem of Descartes [Obr63, Satz 13.2] states that the number of positive zeros of an algebraic polynomial with real coefficients is equal to the number of sign changes in the coefficient sequence or less than it by an even number. In our case, the polynomial Q has no sign changes, and thus all real zeros of the polynomial Q have to be negative. In order to examine the dependency of the non-negativity of the coefficients of Q on the complex zero pairs, we generalize the observations in [Bri85].

Lemma 3.1. *Let Q be a monic polynomial with real coefficients corresponding to the zero set $\{\beta_1, \dots, \beta_{N-1}\}$. Assume that $(\beta_{N-2}, \beta_{N-1})$ is a conjugated zero pair, and define $\sigma_n := (-1)^n S_n(\beta_1, \dots, \beta_{N-3})$ for every $n \in \mathbb{Z}$. Then Q has only non-negative coefficients if and only if β_{N-1} fulfils*

$$\sigma_{n-2} |\beta_{N-1}|^2 - 2\sigma_{n-1} \Re \beta_{N-1} + \sigma_n \geq 0 \quad (n = 0, \dots, N-1). \quad (2)$$

Proof. Since the zeros β_{N-2} and β_{N-1} form a conjugated pair, the monic polynomial Q can be written as

$$Q(z) = (z - \beta_{N-1}) (z - \bar{\beta}_{N-1}) \prod_{j=1}^{N-3} (z - \beta_j).$$

Observing that the product over the first $N-3$ linear factors is itself a monic polynomial, we can again apply Vieta's formulae and obtain

$$\begin{aligned} Q(z) &= \left(z^2 - 2 \Re \beta_{N-1} z + |\beta_{N-1}|^2 \right) \left(\sum_{n=0}^{N-3} \sigma_n z^{N-3-n} \right) \\ &= \sum_{n=0}^{N-1} \left(\sigma_n - 2\sigma_{n-1} \Re \beta_{N-1} + \sigma_{n-2} |\beta_{N-1}|^2 \right) z^{N-1-n}, \end{aligned}$$

which completes the proof. \square

Remark 3.2. Each of the non-negativity constraints (2) describes a certain disc on the Riemann sphere. This allows us to simplify the corresponding inequalities and to interpret them geometrically. For example, if all zeros $\beta_1, \dots, \beta_{N-3}$ have a negative real part, one can show that the zero pair $(\beta_{N-1}, \overline{\beta_{N-1}}^{-1})$ has to lie in the closed half plane left of the imaginary axis through $\sigma_{1/2}$ and, moreover, on or outside the circles with centre $\sigma_{n-1}/\sigma_{n-2}$ and radius

$$\frac{\sqrt{\sigma_{n-1}^2 - \sigma_n \sigma_{n-2}}}{\sigma_{n-2}} \quad (n = 2, \dots, N-2)$$

whenever the radius exists. Indeed (2) implies $\Re \beta_{N-1} \leq \sigma_{1/2}$ for $n = 1$ and

$$\left| \beta_{N-1} - \frac{\sigma_{n-1}}{\sigma_{n-2}} \right|^2 \geq \frac{\sigma_{n-1}^2 - \sigma_n \sigma_{n-2}}{\sigma_{n-2}^2}$$

for $n = 2, \dots, N-2$. This specific behavior is a complex version of the findings by Briggs in [Bri85, Section 7]. \circ

4. Non-negative ambiguities of the phase retrieval problem

Based on our findings about the non-negativity of the coefficients of an algebraic polynomial, we now investigate the non-negativity of the non-trivial solutions x in Theorem 2.2, which can be constructed by reflecting some of the corresponding zeros β_j at the unit circle. First we show that, in the worst case, the additional non-negativity constraint cannot reduce the set of non-trivial solutions at all.

Proposition 4.1. *Let x be a real-valued discrete-time signal with finite support. If the corresponding zero set $\{\beta_1, \dots, \beta_{N-1}\}$ is contained in the left half plane, i.e. $\Re \beta_j < 0$ for all $j = 1, \dots, N-1$, then all occurring real-valued non-trivial ambiguities of the corresponding phase retrieval problem are non-negative.*

Proof. Using Theorem 2.2, we can generate all real-valued non-trivial ambiguities of the phase retrieval problem to recover x by reflecting a subset of the real zeros β_j and

conjugate zero pairs $(\beta_j, \bar{\beta}_j)$ at the unit circle. Since all zeros β_j and hence their reflections $\bar{\beta}_j^{-1}$ have a negative real part, the corresponding linear factors

$$e^{-i\omega} - \beta_j \quad \text{and} \quad (e^{-i\omega} - \beta_j)(e^{-i\omega} - \bar{\beta}_j) = e^{-2i\omega} - 2\Re[\beta_j]e^{-i\omega} + |\beta_j|^2$$

of the real zeros β_j and conjugate zero pairs $(\beta_j, \bar{\beta}_j)$ in (1) have only non-negative coefficients. Thus, all possible non-trivial solutions have only non-negative components since a product of polynomials with non-negative coefficients has again non-negative coefficients. \square

Besides this observation, we can exploit Lemma 3.1 to construct phase retrieval problems with a specific number of non-negative non-trivial solutions.

Example 4.2. We try to construct a phase retrieval problem with at least one non-negative solution x by selecting the free conjugate zero pair (β_4, β_5) of the corresponding zero set

$$\Lambda := \left\{ -\frac{3}{2}, -1 + i, -1 - i, \beta_4, \beta_5 \right\}$$

appropriately. Since the reflection of the complete corresponding zero set leads to the reflection of the original signal, all further non-trivial solutions $y_1, y_2,$ and y_3 according to Theorem 2.2 are given by the zero sets

$$\begin{aligned} M_1 &:= \left\{ -\frac{2}{3}, -1 + i, -1 - i, \beta_4, \beta_5 \right\}, & M_2 &:= \left\{ -\frac{3}{2}, -\frac{1}{2}(1 + i), -\frac{1}{2}(1 - i), \beta_4, \beta_5 \right\}, \\ & & \text{and} & & M_3 &:= \left\{ -\frac{2}{3}, -\frac{1}{2}(1 + i), -\frac{1}{2}(1 - i), \beta_4, \beta_5 \right\}, \end{aligned}$$

respectively.

The non-negativity constraints in Lemma 3.1 for these zero sets are visualized in Figure 1 on the following page. More detailed, the signal x with zero set Λ has only non-negative components if and only if the zero pair (β_4, β_5) lies in the half plane left from the imaginary axis through $7/4$ and outside the circles with centres $7/2, 10/7,$ and $3/5$ in the complex plane and radii $\sqrt{29}/2, \sqrt{58}/7,$ and $3/5$ respectively as discussed in Remark 3.2. The intersection of the half plane and the complements of the three discs is shown in Figure 1(a). The non-negativity constraints for the remaining sets $M_1, M_2,$ and M_3 can be determined analogously. Choosing (β_4, β_5) in one or more intersections, we can thus ensure the non-negativity for certain non-trivial solutions and can directly influence the number of non-negative solutions. For instance, if we choose

$$\beta_4 := \frac{3}{4} + i \quad \text{and} \quad \beta_5 := \frac{3}{4} - i,$$

then the non-negativity constraints for $\Lambda, M_1,$ and M_3 are fulfilled, which means that the phase retrieval problem to recover x has two further non-negative non-trivial ambiguities. The corresponding signals and Fourier intensities are shown in Figure 1(b) and 1(c). \circ

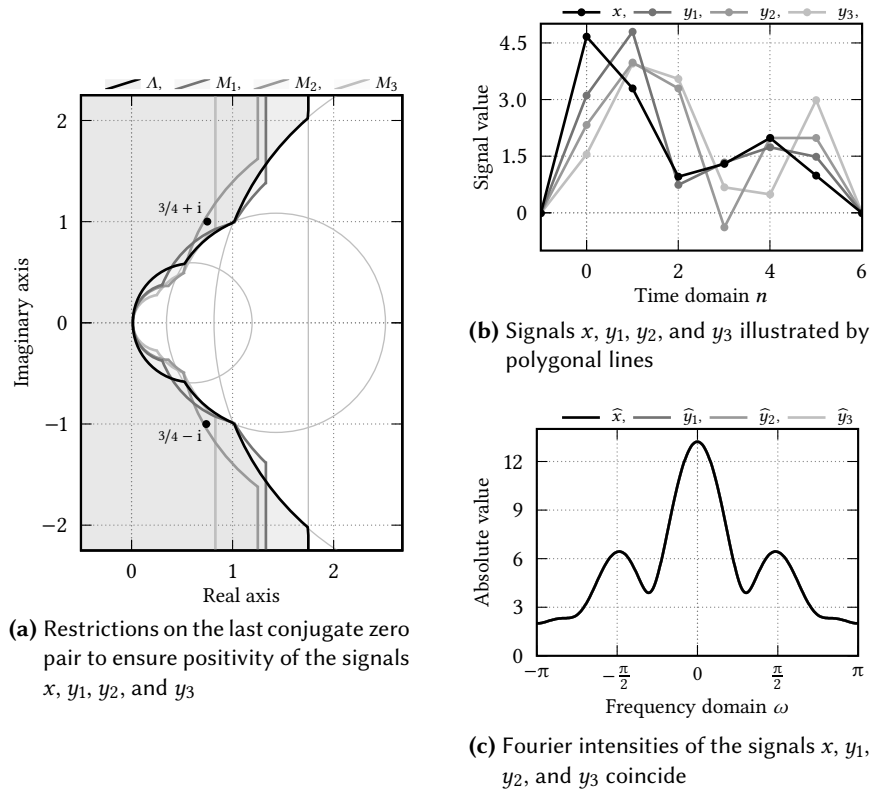


Figure 1: Restriction on the last zero pair in order to ensure non-negativity of different non-trivial ambiguities

5. Uniqueness and ambiguousness under non-negativity constraints

Looking back at Example 4.2, it seems that the non-negativity usually cannot reduce the number of arising non-trivial ambiguities. However, the situation dramatically depends on the fixed zeros $\beta_1, \dots, \beta_{N-3}$ in Lemma 3.1. Although we cannot see the efficiency of the non-negativity constraint directly, we can nevertheless use our findings to show that neither uniqueness nor ambiguousness under the non-negativity constraint are rare exceptions. For this, we exploit that the non-trivial solutions continuously depend on their corresponding zero sets, and vice versa.

Lemma 5.1. *Let x be a discrete-time signal with support $\{0, \dots, N-1\}$ of length N and corresponding zero set $\{\beta_1, \dots, \beta_{N-1}\}$. For every sufficiently small number $\varepsilon > 0$, there exists a number $\delta > 0$ such that the corresponding zeros $\beta_1, \dots, \beta_{N-1}$ of every signal \check{x} with support $\{0, \dots, N-1\}$ of length N and $|\check{x}[n] - x[n]| \leq \delta$ for n from 0 to $N-1$ can be*

ordered in a way that

$$|\check{\beta}_j - \beta_j| \leq \varepsilon$$

for j from 1 to $N - 1$.

Proof. Based on the real-valued signal x , we consider the monic polynomial

$$P(z) = \frac{1}{x[N-1]} \sum_{n=0}^{N-1} x[n] z^n,$$

whose roots coincide with the zero set $\{\beta_1, \dots, \beta_{N-1}\}$. In the following, we denote the coefficients of P by $c_n := x[n]/x[N-1]$. Using the continuity of roots theorem, see [Ort72, Theorem 3.1.1], we find, for every sufficiently small number $\varepsilon > 0$, a number $\eta > 0$ such that the zeros $\check{\beta}_j$ of all monic polynomials

$$Q(z) := z^{N-1} + \check{c}_{N-2} z^{N-2} + \dots + \check{c}_0$$

with $|\check{c}_n - c_n| \leq \eta$ for n from 0 to $N - 2$ can be ordered in a way that

$$|\check{\beta}_j - \beta_j| \leq \varepsilon$$

for j from 1 to $N - 1$.

If we identify x with an N -dimensional vector, then the continuous mapping between the components $x[n]$ and the coefficients c_n is given by

$$(x[0], \dots, x[N-1]) \mapsto \left(\frac{x[0]}{x[N-1]}, \dots, \frac{x[N-2]}{x[N-1]} \right).$$

Hence, for every sufficiently small number $\eta > 0$, there exists a number $\delta > 0$ such that the components of the image of every vector \check{x} in \mathbb{R}^N with $|\check{x}[n] - x[n]| \leq \delta$ for n from 0 to $N - 1$ satisfy

$$\left| \frac{\check{x}[n]}{\check{x}[N-1]} - \frac{x[n]}{x[N-1]} \right| \leq \eta \quad \text{or} \quad |\check{c}_n - c_n| \leq \eta$$

for n from 0 to $N - 2$. In order to avoid that $\check{x}[N-1]$ becomes zero, we assume without loss of generality that $\delta < x[N-1]$. Interpreting the vector \check{x} as discrete-time signal with support $\{0, \dots, N-1\}$ and combining both constructions yield the assertion. \square

Remark 5.2. If we consider the discrete-time signals \check{x} in Lemma 5.1 as N -dimensional vectors, these signals form a closed ball with respect to the maximum norm. Moreover, we can extend this ball to a cone since the multiplication of a signal with a positive real constant does not change the corresponding zero set. By construction, the resulting cone cannot be contained in a set with zero Lebesgue measure. Consequently, this cone is an unbounded set with infinite measure. \circ

Applying Lemma 5.1 to all possible non-trivial ambiguities in Theorem 2.2, we can conclude that the occurring ambiguities continuously depend on the original signal x . In other words, for all signals \check{x} in a small neighbourhood around x , the corresponding phase retrieval problem to recover \check{x} has the same solution behaviour as for x . Moreover, this observation also holds for the number of non-negative solutions if the corresponding zero set of x fulfils some further assumptions. In the next two lemmata, we construct such signals.

Lemma 5.3. *For every $N \in \mathbb{N}$, there exists a signal x with support $\{0, \dots, N-1\}$ and positive components $x[n]$ for n from 0 to $N-1$ such that the phase retrieval problem to recover the signal x has exactly 2^{N-2} non-trivial solutions satisfying the same assumptions.*

Proof. We consider a signal x with $N-1$ distinct real corresponding zeros fulfilling $\beta_j < -1$. As a consequence, the signals in Theorem 2.2 differ up to the trivial reflection ambiguity. Choosing $n_0 = 0$, the phase retrieval problem to recover x thus has 2^{N-2} non-trivially different solutions with support $\{0, \dots, N-1\}$. The positivity of the components immediately follows from Proposition 4.1. \square

Lemma 5.4. *For every $N > 3$, there exists a signal x with support $\{0, \dots, N-1\}$, positive components $x[n]$ for n from 0 to $N-1$, and distinct zeros $\{\beta_1, \dots, \beta_{N-1}\}$ lying not on the unit circle such that the phase retrieval problem to recover the signal x is uniquely solvable up to reflection.*

Proof. Using the approach in Example 4.2, we choose distinct zeros $\beta_1, \dots, \beta_{N-3}$ with $\Re\beta_j < 1$ and extend this set by selecting an appropriate conjugate zero pair $(\beta_{N-2}, \beta_{N-1})$. Since the fixed zeros $\beta_1, \dots, \beta_{N-3}$ lie in the left half plane, we can apply the slightly simpler constraints in Remark 3.2 to ensure the non-negativity of the corresponding signal x . In this manner, $(\beta_{N-2}, \beta_{N-1})$ has to lie in the half plane left of the imaginary axis through $\sigma_{1/2}$, which means that

$$\Re\beta_{N-1} \leq -\frac{1}{2} (\Re\beta_1 + \dots + \Re\beta_{N-3}). \quad (3)$$

By replacing a subset of zeros β_j by their reflections at the unit circle, we obtain the non-negativity constraints for the remaining ambiguities in Theorem 2.2 analogously.

Since $\Re\beta_j < -1$ and thus $\Re\bar{\beta}_j^{-1} > -1$ for j from 1 to $N-3$, the reflection of some zeros at the unit circle leads to a strictly smaller right-hand side of (3). Consequently, we can choose $(\beta_{N-2}, \beta_{N-1})$ so that the zero set $\{\beta_1, \dots, \beta_{N-1}\}$ of the signal x satisfies the non-negativity condition in (3) strictly, and that the zero sets of the remaining non-trivial ambiguities violate this condition. Figuratively, the zeros β_{N-2} and β_{N-1} have to lie in an appropriately small band in the complex plane, cf. Figure 1(a). If we further ensure

that the conjugate zero pair $(\beta_{N-2}, \beta_{N-1})$ strictly lies outside the discs in Remark 3.2, the constructed signal x only possesses positive coefficients $x[n]$ for n from 0 to $N - 1$ as desired. \square

We combine our findings in this section and finally show that neither the uniqueness nor the ambiguousness under the non-negativity constraint is a rare exception. Hence, the assumed non-negativity of a discrete-time signal can be used to enforce the uniqueness of the corresponding phase retrieval problem, but unfortunately not for every signal.

Theorem 5.5. *The set of real-valued discrete-time signals with support $\{0, \dots, N - 1\}$ of length $N > 0$ that can be recovered uniquely up to reflection as well as the set of signals that cannot be recovered uniquely from their Fourier intensities employing the non-negativity constraint are both unbounded sets containing a cone of infinite Lebesgue measure.*

Proof. In Lemma 5.3 and Lemma 5.4, we have constructed signals x so that the corresponding phase retrieval problem has either exactly 2^{N-2} non-trivial ambiguities or is uniquely solvable by choosing the corresponding zero set $\{\beta_1, \dots, \beta_{N-1}\}$ explicitly. Since the non-negativity constraints in Lemma 3.1 and in Remark 3.2 continuously depend on the zeros β_j , there exists a small neighbourhood U_B with respect to the maximum norm around the chosen set $B := \{\beta_1, \dots, \beta_{N-1}\}$ such that the corresponding zero sets $\check{B} \in U_B$ satisfy the same inequalities (2) as B . Now, Lemma 5.1 implies the existence of a small neighbourhood U_x around the constructed signal x so that the corresponding phase retrieval problems have 2^{N-2} non-trivial solutions or are uniquely solvable respectively. Extending the ball U_x to a cone as discussed in Remark 5.2 leads to the assertion. \square

Acknowledgements

I gratefully acknowledge the funding of this work by the DFG in the framework of the SFB 755 ‘Nanoscale photonic imaging’ and of the GRK 2088 ‘Discovering structure in complex data: statistics meets optimization and inverse problems.’

References

- [BBCE09] BALAN, Radu ; BODMANN, Bernhard G. ; CASAZZA, Peter G. ; EDIDIN, Dan: Painless reconstruction from magnitudes of frame coefficients. In: *Journal of Fourier Analysis and Applications* 15 (2009), No. 4, pp. 488–501
- [BCE06] BALAN, Radu ; CASAZZA, Peter G. ; EDIDIN, Dan: On signal reconstruction without phase. In: *Applied and Computational Harmonic Analysis* 20 (2006), No. 3, pp. 345–356
- [BCM14] BANDEIRA, Afonso S. ; CHEN, Yutong ; MIXON, Dustin G.: Phase retrieval from power spectra of masked signals. In: *Information and Interference: A Journal of the IMA* 3 (2014), June, No. 2, pp. 83–102

-
- [Bei16] BEINERT, Robert: *One-dimensional phase retrieval with additional interference measurements*. April 2016. – Preprint, arXiv:1604.04489v1
- [BFGR76] BURGE, R. E. ; FIDDY, M. A. ; GREENAWAY, A. H. ; ROSS, G.: The phase problem. In: *Proceedings of the Royal Society of London. Series A. Mathematical Physical & Engineering Sciences* 350 (1976), pp. 191–212
- [BH15] BODMANN, Bernhard G. ; HAMMEN, Nathaniel: Stable phase retrieval with low-redundancy frames. In: *Advances in Computational Mathematics* 41 (2015), April, No. 2, pp. 317–331
- [BP15] BEINERT, Robert ; PLONKA, Gerlind: Ambiguities in one-dimensional discrete phase retrieval from Fourier magnitudes. In: *Journal of Fourier Analysis and Applications* 21 (2015), December, No. 6, pp. 1169–1198
- [BP16] BEINERT, Robert ; PLONKA, Gerlind: *Enforcing uniqueness in one-dimensional phase retrieval by additional signal information in time domain*. March 2016. – Preprint, arXiv:1604.04493v1
- [Bri85] BRIGGS, William E.: Zeros and factors of polynomials with positive coefficients and protein-ligand binding. In: *Rocky Mountain Journal of Mathematics* 15 (1985), pp. 75–89
- [BS79] BRUCK, Yu. M. ; SODIN, L. G.: On the ambiguity of the image reconstruction problem. In: *Optics communications* 30 (1979), September, No. 3, pp. 304–308
- [CESV13] CANDÈS, Emmanuel J. ; ELДАР, Yonina C. ; STROHMER, Thomas ; VORONINSKI, Vladislav: Phase retrieval via matrix completion. In: *SIAM Journal on Imaging Sciences* 6 (2013), No. 1, pp. 199–225
- [DF87] DAINTY, J. C. ; FIENUP, J. R.: Phase retrieval and image reconstruction for astronomy. In: STARK, Henry (Ed.): *Image Recovery: Theory and Application*. Orlando (Florida) : Academic Press, 1987, Chapter 7, pp. 231–275
- [Fie78] FIENUP, J. R.: Reconstruction of an object from the modulus of its Fourier transform. In: *Optics Letters* 3 (1978), July, No. 1, pp. 27–29
- [Hau91] HAUPTMAN, Herbert A.: The phase problem of x-ray crystallography. In: *Reports on Progress in Physics* 54 (1991), November, No. 11, pp. 1427–1454
- [KH90a] KIM, Wooshik ; HAYES, Monson H.: Iterative phase retrieval using two Fourier transform intensities. In: *Proceedings : ICASSP 90 : 1990 International Conference on Acoustics, Speech and Signal Processing : April 3–6, 1990* Vol. 3 IEEE Signal Processing Society, 1990, pp. 1563–1566
- [KH90b] KIM, Wooshik ; HAYES, Monson H.: Phase retrieval using two Fourier-transform intensities. In: *Journal of the Optical Society of America A* 7 (1990), March, No. 3, pp. 441–449
- [KH93] KIM, Wooshik ; HAYES, Monson H.: Phase retrieval using a window function. In: *IEEE Transactions on Signal Processing* 41 (1993), March, No. 3, pp. 1409–1412
- [LP14] LOOCK, Stefan ; PLONKA, Gerlind: Phase retrieval for Fresnel measurements using a shearlet sparsity constraint. In: *Inverse Problems* 30 (2014), No. 5, pp. 055005(17)
- [LT08] LANGEMANN, Dirk ; TASCHE, Manfred: Phase reconstruction by a multilevel iteratively regularized Gauss-Newton method. In: *Inverse Problems* 24 (2008), No. 3, pp. 035006(26)
- [Mil90] MILLANE, R. P.: Phase retrieval in crystallography and optics. In: *Journal of the Optical Society of America A* 7 (1990), March, No. 3, pp. 394–411
- [Obr63] OBRESCHKOFF, Nikola: *Verteilung und Berechnung der Nullstellen reeller Polynome*. Berlin : VEB Deutscher Verlag der Wissenschaften, 1963 (Hochschulbücher für Mathematik 55)
- [Ort72] ORTEGA, James M.: *Numerical Analysis : A Second Course*. New York : Academic Press, 1972 (Computer Science and Applied Mathematics)
- [RDN13] RAZ, Oren ; DUDOVICH, Nirit ; NADLER, Boaz: Vectorial phase retrieval of 1-D signals. In: *IEEE Transactions on Signal Processing* 61 (2013), April, No. 7, pp. 1632–1643
-

-
- [SSD⁺06] SEIFERT, Birger ; STOLZ, Heinrich ; DONATELLI, Marco ; LANGEMANN, Dirk ; TASCHE, Manfred: Multilevel Gauss-Newton methods for phase retrieval problems. In: *Journal of Physics. A. Mathematical and General* 39 (2006), No. 16, pp. 4191–4206
- [SSTo4] SEIFERT, Birger ; STOLZ, Heinrich ; TASCHE, Manfred: Nontrivial ambiguities for blind frequency-resolved optical gating and the problem of uniqueness. In: *Journal of the Optical Society of America B* 21 (2004), May, No. 5, pp. 1089–1097
-