**Approximation by $k$-sparse sums of eigenfunctions of linear operators**


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(joint work with Thomas Peter)


In signal analysis, there often is some a priori knowledge about the underlying structure of the wanted signal. Thus, one is faced with the problem of extracting a certain number of parameters from the given signal measurements. Considering for example a structured function of the form

$$f(\omega) = \sum_{j=1}^{k} c_j \, e^{\omega T_j}$$

with complex parameters $c_j$ and $T_j$, $j = 1, \ldots, k$, and assuming that $-\pi < \mathrm{Im}T_1 < \ldots < \mathrm{Im}T_k < \pi$, one aims to reconstruct $c_j$ and $T_j$ from a given small amount of (possibly noisy) measurement values $f(\ell)$. Using Prony's method or one of its stabilized variants, one is able to reconstruct $f$ with only $2k$ function values $f(\ell)$, $\ell = 0, \ldots, 2k-1$. The solution of this problem involves the determination of a so-called "Prony polynomial"

$$\Lambda(z) = \prod_{j=1}^{k} (z - e^{T_j}) = \sum_{\ell=0}^{k} \alpha_\ell z^\ell$$

with $\alpha_k = 1$. Using the structure of $f$, a short computation yields

$$\sum_{\ell=0}^{k} f(\ell + m)\alpha_\ell = 0, \qquad m = 0, 1, \ldots. \tag{1}$$

The homogenous Hankel system (1) provides the coefficients $\alpha_\ell$ of the Prony polynomial $\Lambda(z)$, and the unknown parameters $T_j$ can now be extracted from the zeros of $\Lambda(z)$. Afterwards, the coefficients $c_j$ are obtained by solving a linear system.

In recent years, the Prony method has been successfully applied to different inverse problems as e.g. for analysis of ultrasonic signals or for the approximation of Green functions in quantum chemistry or fluid dynamics, see e.g. [2, 3]. The renaissance of Prony's method originates from some modifications of the corresponding algorithm that considerably stabilize the original approach, [4, 7].

Searching the literature, one finds different further reconstruction methods that are closely related to Prony's method at second glance. In spectral analysis the annihilating filter method is frequently applied. This idea has also been used already long ago for the construction of cyclic codes, [8]. For a given signal $S[n]$, the FIR filter $A[n]$ is called annihilating filter of $S[n]$, if

$$(A * S)(n) = \sum_{j \in \mathbb{Z}} A[j] \, S[n-j] = 0.$$

Using the $z$-transform $A(z) = \sum_{n=0}^{k} A[n] z^{-n}$ and comparing this equation to (1), we observe that $z^k A(z)$ undertakes the task of the Prony-polynomial.

In computer algebra, one is faced with the computation and processing of multivariate polynomials of high order. But if the polynomial $f$ is $k$-sparse, i.e.,

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{k} c_j x_1^{d_{j_1}} x_2^{d_{j_2}} \cdots x_n^{d_{j_n}}$$

with $c_1, \ldots, c_k \in \mathbb{C}$ and with $k$ pairwise different vectors $(d_{j_1}, \ldots, d_{j_n}) \in \mathbb{N}^n$, the polynomial can be completely recovered using only $2k$ suitably chosen function values. Here again, the number of needed evaluations does not depend on the degree of the polynomial $f$ but on the number $k$ of active terms. The corresponding algorithm goes back to Ben-Or and Tiwari [1], and has recently been shown to be closely related to the Prony method. In [6], we considered the function reconstruction problem for sparse Legendre expansions of order $N$ of the form

$$f(x) = \sum_{j=1}^{k} c_j P_{n_j}(x)$$

with $0 \leq n_1 < n_2 \ldots < n_k = N$, where $k \ll N$, aiming at a generalization of Prony's method for this case. We succeeded to derive a reconstruction algorithm involving the function and derivative values $f^{(\ell)}(1)$, $\ell = 0, \ldots, 2k-1$. The reconstruction is based on special properties of Legendre polynomials and does not provide an idea for further generalization of the method to other orthogonal polynomial bases or to other function systems apart from exponentials and monomials.

Just recently, we developed a new perception of Prony's method based on eigenfunctions of linear operators, see [5]. This new insight gives us a tool for unification of all Prony-like methods on the one hand and for an essential generalization of the Prony approach on the other hand. This generalization will open a much broader field of applications of the method.

## References


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