Necessary and Sufficient Conditions for Orthonormality of Scaling Vectors

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Abstract. The paper studies necessary and sufficient orthonormality conditions for a scaling vector in terms of its two-scale symbol and its corresponding transfer operator. In particular, it is shown that the conditions of Hogan [10] for the two-scale symbol and the criteria of Shen [21] for the transfer operator are equivalent.

1. Introduction

In this paper we shall discuss orthonormality of compactly supported scaling vectors. These are solutions of functional equations of type

$$\mathbf{\Phi}(x) = \sum_{l=0}^{N} \mathbf{P}_l \, \mathbf{\Phi}(2x - l) \tag{1}$$

with real $r \times r$ coefficient matrices $\mathbf{P}_l$ ($r \in \mathbb{N}$, $r \geq 1$) and with an r-dimensional function vector $\mathbf{\Phi} = (\phi_1, \ldots, \phi_r)^T$. Equations of the form (1) are called *matrix refinement equations.*

If additionally, the functions $\phi_\nu(\cdot - l)$ ($l \in \mathbb{Z}$, $\nu = 1, \ldots, r$) form an orthonormal or an L^2-stable basis of their span, then $\mathbf{\Phi}$ is called *multi-scaling function.* In this case, $\mathbf{\Phi}$ can generate a multiresolution analysis with multiplicity r (see [7]). Once, an MRA generated by an orthonormal multi-scaling function $\mathbf{\Phi}$ is given, the construction of an orthonormal multiwavelet $\mathbf{\Psi} = (\psi_1, \ldots, \psi_r)^T$ can be reduced to a problem of matrix extension, as described in [17].
By Fourier transform of (1), we have

$$\hat{\mathbf{\Phi}}(\omega) = \mathbf{P}(\frac{\omega}{2}) \, \hat{\mathbf{\Phi}}(\frac{\omega}{2}), \tag{2}$$

Proceedings of Mannheim Conference 1996
G. Nürnberger, J. W. Schmidt, and G. Walz (eds.), pp. 1–14.

ISBN x-xxxxx-xxx-x.

where $\hat{\mathbf{\Phi}}$ is taken componentwisely, *i.e.*, $\hat{\mathbf{\Phi}}(\omega) := (\hat{\phi}_1(\omega), \ldots, \hat{\phi}_r(\omega))^T$ with $\hat{\phi}_\nu(\omega) := \int_{-\infty}^{\infty} \phi_\nu(x)\, e^{-i\omega x}\, \mathrm{d}x$ $(\nu = 1, \ldots, r)$, and where

$$\mathbf{P}(\omega) := \sum_{l=0}^{N} \mathbf{P}_l\, e^{-i\omega l} \tag{3}$$

denotes the *two-scale symbol* of $\mathbf{\Phi}$.

Hence, we are faced with the problem of how the orthonormality or L^2-stability condition for a solution vector $\mathbf{\Phi}$ can be ensured by appropriate choice of $\mathbf{P}(\omega)$.

As in the scalar case $(r = 1)$, we observe three different methods to express necessary and sufficient stability (orthonormality) conditions in terms of the two-scale symbol $\mathbf{P}(\omega)$.

The first method is based on the so-called *transfer operator* T associated with $\mathbf{P}(\omega)$. Under certain basic conditions on $\mathbf{P}(\omega)$, L^2-stability (and orthonomality, respectively) of $\mathbf{\Phi}$ can be ensured if the transfer operator T associated with $\mathbf{P}(\omega)$ satisfies special spectral conditions. This method can also be applied in the multivariate setting (see [21]). In the meantime, it turned out that the basic conditions assumed by Shen [21] are necessary for stability of $\mathbf{\Phi}$ (see [4,11,14]).

In order to handle the transfer operator T in practice, one has to use its representing matrix, which in fact can be given explicitly in terms of Kronecker products of coefficient matrices $\mathbf{P}_n$ (see [15,20]). The resulting conditions, which are spectral conditions to the representing matrix, can be seen as generalization of Lawton's criteria for scaling functions (see [5,16]).

Second, there are some successful attempts to find necessary and sufficient conditions directly in terms of the trigonometric polynomial matrix $\mathbf{P}(\omega)$ in order to ensure orthonomality, stability or even local linear independence of the solution vector $\mathbf{\Phi}$ (see [10,22]). These results generalize the well-known Cohen criteria [1] and the conditions of Jia and Wang [13]. But this time, the conditions for the two-scale symbol $\mathbf{P}(\omega)$ are much more complicated, since products of matrix polynomials generally do no commute. Moreover, one is faced with a problem which need not to be handled in the scalar case, namely, of how to ensure the algebraic linear independence of the components ϕ_ν $(\nu = 1, \ldots, r)$ of $\mathbf{\Phi}$ and their translates in terms of $\mathbf{P}(\omega)$.

Third, we want to mention that the stability of scaling vectors is closely related with the convergence of corresponding subdivision schemes and cascade algorithms. In fact, the convergence of the stationary subdivision scheme can be taken as a criteria for stability of $\mathbf{\Phi}$ in $L^p(\mathbb{R})$. This subject is addressed in [3,4,12]. In particular, relations between spectral conditions of the transfer operator and the convergence of the cascade algorithm are considered in [21].

We are especially interested in the first two methods. The purpose of this paper is to study the relation between the spectral properties of the transfer operator T and the properties of the two-scale symbol $\mathbf{P}(\omega)$ in the case of orthonormal scaling vectors. In particular, we shall show that the conditions of Hogan [10] and Shen [21] are indeed equivalent.

2. Basic conditions and uniqueness of $\mathbf{\Phi}$

In this section we want to provide some necessary conditions for orthonormality of a compactly supported solution vector $\mathbf{\Phi}$ of a matrix refinement equation (1). These *basic* conditions will also ensure that $\mathbf{\Phi}$ is unique, and moreover that the components of $\mathbf{\Phi}$ are contained in $L^2(\mathbb{R})$.

We say that a function vector $\mathbf{\Phi}$ (with $\phi_\nu \in L^2(\mathbb{R})$) is *$L^2$-stable* if the integer translates of ϕ_ν are algebraically linearly independent and if there are constants $0 < A \leq B < \infty$, such that

$$A \sum_{l=-\infty}^{\infty} \mathbf{c}_l^T \overline{\mathbf{c}}_l \leq \| \sum_{l=-\infty}^{\infty} \mathbf{c}_l^T \, \mathbf{\Phi}(\cdot - l) \|_{L^2}^2 \leq B \sum_{l=-\infty}^{\infty} \mathbf{c}_l^T \overline{\mathbf{c}}_l \tag{4}$$

for any vector sequence $\{\mathbf{c}_l\}_{l \in \mathbb{Z}} \in l_2^r$. Here l_2^r denotes the set of sequences of vectors $(\mathbf{c}_l)_{l\in\mathbb{Z}}$ ($\mathbf{c}_l \in \mathbb{C}^r$) with $\sum_{l=-\infty}^{\infty} \mathbf{c}_l^T \overline{\mathbf{c}_l} < \infty$. $\mathbf{\Phi}$ is called *orthonormal* if (4) is satisfied with $A = B = 1$, in other words, if

$$\langle \phi_\mu, \, \phi_\nu(\cdot - n) \rangle_{L^2} = \delta_{0,n} \, \delta_{\mu,\nu}. \tag{5}$$

Introducing the autocorrelation symbol

$$\begin{aligned} \mathbf{\Omega}(\omega) &:= \sum_{n \in \mathbb{Z}} \left(\langle \phi_\mu, \, \phi_\nu(\cdot - n) \rangle_{L^2} \right)_{\mu,\nu=1}^r e^{-i\omega n} \\ &= \sum_{n \in \mathbb{Z}} \left(\int_{-\infty}^{\infty} \mathbf{\Phi}(x) \, \mathbf{\Phi}(x-n)^\star \, \mathrm{d}x \right) e^{-i\omega n} \end{aligned}$$

(with $\mathbf{\Phi}(x)^\star := \overline{\mathbf{\Phi}(x)^T}$) we simply observe that (5) is equivalent with $\mathbf{\Omega}(\omega) = \mathbf{I}$, where $\mathbf{I}$ denotes the unit matrix of size r. Further, the stability condition (4) implies that the autocorrelation symbol is strictly positive definite for all $\omega \in \mathbb{R}$ (see [7]).

From

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathbf{\Phi}(x) \, \mathbf{\Phi}(x-n)^\star \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\mathbf{\Phi}}(\omega) \, \hat{\mathbf{\Phi}}(\omega)^\star \, e^{in\omega} \, \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{in\omega} \sum_{l \in \mathbb{Z}} \hat{\mathbf{\Phi}}(\omega + 2\pi l) \, \hat{\mathbf{\Phi}}(\omega + 2\pi l)^\star \, \mathrm{d}\omega \end{aligned}$$

it follows that

$$\mathbf{\Omega}(\omega) = \sum_{l \in \mathbb{Z}} \hat{\mathbf{\Phi}}(\omega + 2\pi l) \, \hat{\mathbf{\Phi}}(\omega + 2\pi l)^\star \qquad a.e.. \tag{6}$$

Substituting (2) leads to

$$\mathbf{\Omega}(\omega) = \sum_{l\in\mathbb{Z}} \mathbf{P}(\frac{\omega}{2}+\pi l)\, \hat{\mathbf{\Phi}}(\frac{\omega}{2}+\pi l)\, \hat{\mathbf{\Phi}}(\frac{\omega}{2}+\pi l)^{\star}\, \mathbf{P}(\frac{\omega}{2}+\pi l)^{\star}.$$

Splitting the sum into even and odd l, it follows that

$$\mathbf{\Omega}(\omega) = \mathbf{P}(\frac{\omega}{2})\, \mathbf{\Omega}(\frac{\omega}{2})\, \mathbf{P}(\frac{\omega}{2})^{\star} + \mathbf{P}(\frac{\omega}{2}+\pi)\, \mathbf{\Omega}(\frac{\omega}{2}+\pi)\, \mathbf{P}(\frac{\omega}{2}+\pi)^{\star} \qquad a.e..$$

Since $\mathbf{P}(\omega)$ and $\mathbf{\Omega}(\omega)$ are assumed to be matrices of trigonometric polynomials, the a.e. can be droped. In particular, for orthonomal L^2-solutions of (1), the condition

$$\mathbf{I} = \mathbf{P}(\frac{\omega}{2})\, \mathbf{P}(\frac{\omega}{2})^{\star} + \mathbf{P}(\frac{\omega}{2}+\pi)\, \mathbf{P}(\frac{\omega}{2}+\pi)^{\star} \tag{7}$$

is necessarily satisfied.

Let $\mathbb{H} = \mathbb{H}_N$ be the space of trigonometric polynomials of degree at most N, *i.e.*, the elements of $\mathbb{H}$ are of the form $h(\omega) = \sum_{n=-N}^{N} h_n\, e^{-i\omega n}$ $(h_n \in \mathbb{C})$. We introduce the following *transfer operator* $T : \mathbb{H}^{r\times r} \to \mathbb{H}^{r\times r}$,

$$T\,\mathbf{H}(\omega) := \mathbf{P}(\frac{\omega}{2})\, \mathbf{H}(\frac{\omega}{2})\, \mathbf{P}(\frac{\omega}{2})^{\star} + \mathbf{P}(\frac{\omega}{2}+\pi)\, \mathbf{H}(\frac{\omega}{2}+\pi)\, \mathbf{P}(\frac{\omega}{2}+\pi)^{\star},$$

acting on $(r\times r)$-matrices $\mathbf{H}(\omega)$ with elements of $\mathbb{H}$ as entries. Observe that the autocorrelation symbol $\mathbf{\Omega}(\omega)$ is an eigenmatrix of the transfer operator T corresponding to the eigenvalue 1.

For a square matrix $\mathbf{M}$ (or a linear operator) let us introduce the following

Condition E. *The spectral radius of $\boldsymbol{M}$ is less than or equal to 1, i.e. $\rho(\boldsymbol{M}) \le 1$, and 1 is the only eigenvalue of $\boldsymbol{M}$ on the unit circle. Moreover, 1 is a simple eigenvalue.*

Assuming that the components of a solution vector $\mathbf{\Phi}$ of (1) are compactly supported and in $L^2(\mathbb{R})$, it necessarily follows that they are also contained in $L^1(\mathbb{R})$. As shown in [4,11], we have:

Proposition 1. *Let $\mathbf{\Phi}$ be a stable L_1-solution vector of (1). Then for the corresponding symbol $\boldsymbol{P}(\omega)$ we have:*

a) *$\boldsymbol{P}(0)$ satisfies Condition E.*

b) *There exists a nonzero vector $\boldsymbol{y} \in \mathbb{R}^r$ such that $\boldsymbol{y}^T \boldsymbol{P}(0) = \boldsymbol{y}^T$ and $\boldsymbol{y}^T \boldsymbol{P}(\pi) = \boldsymbol{0}^T$. Equivalently, the solution vector $\mathbf{\Phi}$ provides approximation order 1, i.e., we have*

$$\boldsymbol{y}^T \sum_{l=-\infty}^{\infty} \mathbf{\Phi}(\cdot - l) = c,$$

where $\boldsymbol{y}$ is a left eigenvector of $\boldsymbol{P}(0)$ to the eigenvalue 1, and c is a nonvanishing constant.

The necessary conditions of Proposition 1 for $\mathbf{P}(\omega)$ are called *basic conditions.*

In the rest of the paper, we want to assume that the basic conditions and the orthonomality condition (7) are satisfied for the two-scale symbol $\mathbf{P}(\omega)$. Indeed, these assumptions already imply the uniqueness of a solution vector of compactly supported L^2-functions. Using the results of Jiang and Shen [14], we find:

Proposition 2. *Let $\boldsymbol{P}(\omega)$ be of the form (3) satisfying the basic conditions of Proposition 1, and let $\mathbf{a}$ be a right eigenvector of $\boldsymbol{P}(0)$ corresponding to the eigenvalue 1. Then (1) provides a compactly supported distribution solution $\boldsymbol{\Phi}$, where*

$$\hat{\boldsymbol{\Phi}}(\omega) := \lim_{L\to\infty} \prod_{j=1}^{L} \boldsymbol{P}\left(\frac{\omega}{2^j}\right) \mathbf{a}. \tag{8}$$

This solution vector $\boldsymbol{\Phi}$ is unique up to multiplication with a constant.

Note that this result is nontrivial; it is based on the observation that the growth of $\hat{\phi}_1, \ldots, \hat{\phi}_r$ is at most polynomial on $\mathbb{R}$ (see also [9]). Proposition 2 can be seen as a generalization of an analogous result for scalar refinement equations by Deslauries and Dubuc [6]. Let us mention, that the pointwise convergence of the infinite product in (8) can even be shown, if only $\rho(\mathbf{P}(0)) < 2$ is satisfied (see [2,8]).

Further, we find:

Proposition 3. *Let $\boldsymbol{P}(\omega)$ be of the form (3), satisfying the basic conditions of Proposition 1 and the orthonormality condition (7). Further, let $\mathbf{a}$ be a right eigenvector of $\boldsymbol{P}(0)$ to the eigenvalue 1. Then the function vector $\hat{\boldsymbol{\Phi}}$ given in (8) is contained in $L^2(\mathbb{R}^r)$.*

Proof: We introduce $\boldsymbol{\Pi}_n(\omega) := \prod_{j=1}^{n} \mathbf{P}(\frac{\omega}{2^j})$ and $\tilde{\boldsymbol{\Pi}}_n(\omega) := \chi_{[-\pi,\pi]}(2^n\omega)\,\boldsymbol{\Pi}_n(\omega)$, where $\chi_{[-\pi,\pi]}$ denotes the characteristic function over $[-\pi,\pi]$. Obviously, $\tilde{\boldsymbol{\Pi}}_n(\omega)$ converges pointwise to $\boldsymbol{\Pi}(\omega) := \prod_{j=1}^{\infty} \mathbf{P}(\frac{\omega}{2^j})$. By (7), we find that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\boldsymbol{\Pi}}_n(\omega)\,\tilde{\boldsymbol{\Pi}}_n(\omega)^\star \mathrm{d}\omega = \frac{1}{2\pi}\int_{-2^n\pi}^{2^n\pi} \boldsymbol{\Pi}_n(\omega)\,\boldsymbol{\Pi}_n(\omega)^\star\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{0}^{2^{n+1}\pi} \boldsymbol{\Pi}_n(\omega)\,\boldsymbol{\Pi}_n(\omega)^\star\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{0}^{2^n\pi} \boldsymbol{\Pi}_{n-1}(\omega)\,[\mathbf{P}(\frac{\omega}{2^n})\,\mathbf{P}(\frac{\omega}{2^n})^\star + \mathbf{P}(\frac{\omega}{2^n}+\pi)\,\mathbf{P}(\frac{\omega}{2^n}+\pi)^\star]\boldsymbol{\Pi}_{n-1}(\omega)^\star\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{0}^{2^n\pi} \boldsymbol{\Pi}_{n-1}(\omega)\,\boldsymbol{\Pi}_{n-1}(\omega)^\star\,\mathrm{d}\omega = \ldots \\
&= \frac{1}{2\pi}\int_{0}^{4\pi} \boldsymbol{\Pi}_1(\omega)\,\boldsymbol{\Pi}_1(\omega)^\star\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{0}^{2\pi} \mathbf{I}\,\mathrm{d}\omega = \mathbf{I}
\end{aligned}$$

for all $n \in \mathbb{N}$. Let now $\|\cdot\|$ denote the spectral matrix norm and the Euklidian vector norm, respectively. Using the Lemma of Fatou, it follows that

$$\begin{aligned}
\|\boldsymbol{\Pi}\|^2_{L^2(\mathbb{R}^{r\times r})} &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\boldsymbol{\Pi}(\omega)\|^2\,\mathrm{d}\omega \le \limsup_{n\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\tilde{\boldsymbol{\Pi}}_n(\omega)\|^2\,\mathrm{d}\omega \\
&= \limsup_{n\to\infty} \|\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\boldsymbol{\Pi}}_n(\omega)\,\tilde{\boldsymbol{\Pi}}_n(\omega)^\star \mathrm{d}\omega\| = 1.
\end{aligned}$$

Finally, observing that

$$\|\hat{\mathbf{\Phi}}\|_{L^2(\mathbb{R}^r)} := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|\mathbf{\Pi}(\omega)\,\mathbf{a}\|^2\,\mathrm{d}\omega\right)^{1/2} \le \|\mathbf{\Pi}\|_{L^2(\mathbb{R}^{r\times r})}\,\|\mathbf{a}\| = \|\mathbf{a}\| < \infty,$$

the assertion follows. ■

Remark. This result is a direct generalization of the result of Mallat [19]. For the characterization of L^2-solutions of (1) see also [9,14].

3. Algebraic linear independence of scaling vectors

We say that a compactly supported function vector $\mathbf{\Phi}$ is *(algebraically) linearly independent* if for a finite linear combination

$$\sum_{l\in\mathbb{Z}}\sum_{\nu=1}^{r} c_{\nu,l}\,\phi_\nu(\cdot - l) = 0 \quad \Rightarrow \quad c_{\nu,l} = 0 \quad \text{for all} \quad \nu, l.$$

Equivalently, $\mathbf{\Phi}$ is algebraically linearly independent if

$$\mathbf{A}(\omega)^T\,\hat{\mathbf{\Phi}}(\omega) = 0 \quad \Rightarrow \quad \mathbf{A}^T(\omega) = \mathbf{0}^T$$

for arbitrary vectors $\mathbf{A}(\omega)$ of trigonometric polynomials. Note that the linear independence is necessary for L^2-stability or orthonormality of $\mathbf{\Phi}$.

In the scalar case, the problem of linear independence of integer translates of a scaling function $\phi \in L^1(\mathbb{R})$ need not to be handled, since $A(\omega)\,\hat{\phi}(\omega) = 0$ for some trigonometric polynomial $A(\omega)$ would imply that for all $\omega \in \mathbb{R}$ either $A(\omega) = 0$ or $\hat{\phi}(\omega) = 0$. But from $\hat{\phi}(0) \ne 0$ it follows, by continuity, that $\hat{\phi}(\omega) \ne 0$ in a neighborhood of 0. So, $A(\omega) = 0$ for ω in a neighborhood of 0, and hence for all $\omega \in \mathbb{R}$.

For $r > 1$, we need to investigate this problem. First we observe:

Lemma 4. *Assume that the compactly supported function vector $\mathbf{\Phi} \in L^2(\mathbb{R}^r)$ satisfies the Poisson summation formula. Then we have:*
$\mathbf{\Phi}$ is algebraically linearly dependent if and only if its autocorrelation symbol $\mathbf{\Omega}(\omega)$ satisfies

$$\det\,\mathbf{\Omega}(\omega) = 0$$

for all $\omega \in [-\pi, \pi)$.

Proof: The Poisson summation formula implies that the autocorrelation symbol $\mathbf{\Omega}(\omega)$ of $\mathbf{\Phi}$ is of the form (6), where the *a.e.* can be droped.

1. Assume that $\mathbf{\Phi}$ is algebraically linearly dependent, then, by definition, there exists a nontrivial vector of trigonometric polynomials $\mathbf{A}(\omega)$ such that

$$\mathbf{A}(\omega)^T\,\hat{\mathbf{\Phi}}(\omega) = 0$$

for all $\omega \in [-\pi, \pi)$. Hence, we have

$$\mathbf{A}(\omega)^T \sum_{l=-\infty}^{\infty} \hat{\mathbf{\Phi}}(\omega + 2\pi l)\, \hat{\mathbf{\Phi}}(\omega + 2\pi l)^\star\, \overline{\mathbf{A}(\omega)} = 0,$$

that means, $\det \mathbf{\Omega}(\omega) = 0$ for all $l \in \mathbb{Z}$.

2. Assume that $\mathbf{\Omega}(\omega)$ is singular for all $\omega \in [-\pi, \pi)$. Then there exists a vector $\mathbf{A}(\omega)$ of 2π-periodic functions such that

$$\mathbf{A}(\omega)^T\, \mathbf{\Omega}(\omega)\, \overline{\mathbf{A}(\omega)} = \mathbf{A}(\omega)^T \sum_{l=-\infty}^{\infty} \hat{\mathbf{\Phi}}(\omega + 2\pi l)\, \hat{\mathbf{\Phi}}(\omega + 2\pi l)^\star\, \overline{\mathbf{A}(\omega)} = 0.$$

Moreover, since $\mathbf{\Omega}(\omega)$ is a matrix of trigonometric polynomials, we can also find a suitable $\mathbf{A}(\omega)$ with trigonometric polynomials as entries. Observing that each summand $\hat{\mathbf{\Phi}}(\omega+2\pi l)\, \hat{\mathbf{\Phi}}(\omega+2\pi l)^\star$ is positive semidefinite, it follows that $\mathbf{A}(\omega)^T\, \hat{\mathbf{\Phi}}(\omega + 2\pi l) = 0$ for all $l \in \mathbb{Z}$, *i.e.*, $\mathbf{\Phi}$ is linearly dependent. ■

Note that Lemma 4 is not restricted to scaling vectors.

For $r > 1$ the following conditions on the two-scale symbol $\mathbf{P}(\omega)$ imply linear dependence of the solution vector $\mathbf{\Phi}$ of (1):

Theorem 5. *Let $\boldsymbol{P}(\omega)$ be of the form (3), satisfying the basic conditions of Proposition 1 and (7). Let $\mathbf{a}$ be a right eigenvector of $\boldsymbol{P}(0)$ to the eigenvalue 1. Then the following assertions are equivalent:*

(a) *The solution vector $\mathbf{\Phi}$ of (1), determined in (8), is algebraically linearly dependent.*

(b) *There exists an $(r-s)\times r$ matrix $\tilde{\boldsymbol{M}}(\omega)$ of trigonometric polynomials with $\mathrm{rank}(\tilde{\boldsymbol{M}}(0)) = r - s$ such that*

$$\begin{aligned} \tilde{\boldsymbol{M}}(0)\,\mathbf{a} &= \boldsymbol{0}, \\ \tilde{\boldsymbol{M}}(2\omega)\,\boldsymbol{P}(\omega)\,\boldsymbol{M}(\omega) &= \boldsymbol{0} \end{aligned} \tag{9}$$

with zero matrices of suitable size, and where $\boldsymbol{M}(\omega)$ is an $r\times s$ matrix of trigonometric polynomials with $\tilde{\boldsymbol{M}}(\omega)\,\boldsymbol{M}(\omega) = \boldsymbol{0}$.

(c) *There exists a positive semidefinite, hermitian matrix $\boldsymbol{F}(\omega) \in \mathbb{H}^{r\times r}$ with $\det \boldsymbol{F}(\omega) = 0$ for all $\omega \in [-\pi, \pi)$ and satisfying $T\,\boldsymbol{F}(\omega) = \boldsymbol{F}(\omega)$, i.e.,*

$$\boldsymbol{P}(\omega)\,\boldsymbol{F}(\omega)\,\boldsymbol{P}(\omega) + \boldsymbol{P}(\omega+\pi)\,\boldsymbol{F}(\omega+\pi)\,\boldsymbol{P}(\omega+\pi) = \boldsymbol{F}(2\omega). \tag{10}$$

Proof: The equivalence of (a) and (b) was already shown by Hogan [10].

We only need to show the equivalence of (a) and (c).

1. Let $\mathbf{\Phi}$ be linearly dependent. Then, by Lemma 4, its autocorrelation symbol $\mathbf{\Omega}(\omega)$ satisfies $\det \mathbf{\Omega}(\omega) = 0$ for all $\omega \in \mathbb{R}$. Moreover, $\mathbf{\Omega}(\omega)$ is an eigenmatrix of T corresponding to the eigenvalue 1.

2. Assume that $\mathbf{\Phi}$ is linearly independent, and that $\mathbf{F}(\omega)$ is a positive semidefinite, hermitian matrix of trigonometric polynomials satisfying (10) and with $\det \mathbf{F}(\omega) = 0$ for all ω. Then, there exists a nontrivial vector $\mathbf{A}(\omega)$ of trigonometric polynomials such that

$$\mathbf{A}(\omega)^T\, \mathbf{F}(\omega)\, \overline{\mathbf{A}(\omega)} = 0 \quad \text{for all } \omega \in \mathbb{R}\,.$$

Hence, (10) implies that also $\mathbf{A}(\omega)^T\, \mathbf{P}(\frac{\omega}{2})\, \mathbf{F}(\frac{\omega}{2})\, \mathbf{P}(\frac{\omega}{2})^\star\, \overline{\mathbf{A}(\omega)} = 0$ and $\mathbf{A}(\omega)^T\, \mathbf{P}(\frac{\omega}{2} + \pi)\, \mathbf{F}(\frac{\omega}{2} + \pi)\, \mathbf{P}(\frac{\omega}{2} + \pi)^\star\, \overline{\mathbf{A}(\omega)} = 0$. Using the notion $\tilde{\mathbf{\Pi}}_n(\omega) := \chi_{[-\pi,\pi]}(2^{-n}\omega) \prod_{l=1}^{n} \mathbf{P}(\frac{\omega}{2^l})$, it follows, by repeated application of (10), that

$$\mathbf{A}(\omega)^T\, \tilde{\mathbf{\Pi}}_n(\omega)\, \mathbf{F}(\frac{\omega}{2^n})\, \tilde{\mathbf{\Pi}}_n(\omega)^\star\, \overline{\mathbf{A}(\omega)} = 0$$

and finally for $n \to \infty$, for all $\omega \in \mathbb{R}$

$$\mathbf{A}(\omega)^T\, \mathbf{\Pi}(\omega)\, \mathbf{F}(0)\, \mathbf{\Pi}(\omega)^\star\, \overline{\mathbf{A}(\omega)} = 0 \tag{11}$$

with $\mathbf{\Pi}(\omega) = \prod_{l=1}^{\infty} \mathbf{P}(\frac{\omega}{2^l})$. (Observe that, by basic conditions, $\tilde{\mathbf{\Pi}}_n$ converges pointwise to $\mathbf{\Pi}(\omega)$ for all ω.)

3. Let $\mathbf{a}$ be a right eigenvector, and let $\mathbf{y}$ be a left eigenvector of $\mathbf{P}(0)$ to the simple eigenvalue 1, then $\mathbf{y}^T\, \mathbf{a} \neq 0$. We show that $\mathbf{y}^T\, \mathbf{F}(0)\, \overline{\mathbf{y}} = 0$:

We can assume that $\mathbf{F}(0)$ is of the form $c\,\mathbf{a}\mathbf{a}^\star + \mathbf{F}_0(0)$, where c is a suitable nonnegative constant and $\mathbf{F}_0(0)$ is a positive semidefinite matrix satisfying $\mathbf{y}^T\, \mathbf{F}_0(0)\, \overline{\mathbf{y}} = 0$. Then (11) implies that $c\,\mathbf{A}(\omega)^T\, \mathbf{\Pi}(\omega)\, \mathbf{a}\mathbf{a}^\star\, \mathbf{\Pi}(\omega)^\star\, \overline{\mathbf{A}(\omega)} = 0$ for all ω, and hence

$$\begin{aligned} & c\,\mathbf{A}(\omega)^T \left(\sum_{l\in\mathbb{Z}} \mathbf{\Pi}(\omega + 2\pi l)\, \mathbf{a}\mathbf{a}^\star\, \mathbf{\Pi}(\omega + 2\pi l)^\star \right) \overline{\mathbf{A}(\omega)} \\ &= c\,\mathbf{A}(\omega)^T \left(\sum_{l\in\mathbb{Z}} \hat{\mathbf{\Phi}}(\omega + 2\pi l)\, \hat{\mathbf{\Phi}}(\omega + 2\pi l)^\star \right) \overline{\mathbf{A}(\omega)} = c\,\mathbf{A}(\omega)^T\, \mathbf{\Omega}(\omega)\, \overline{\mathbf{A}(\omega)} = 0. \end{aligned}$$

But, since $\mathbf{\Phi}$ is assumed to be linearly independent, its autocorrelation symbol is nonsingular a.e., and hence $c = 0$. So, we find $\mathbf{y}^T\, \mathbf{F}(0)\, \overline{\mathbf{y}} = 0$.

4. We introduce the space $V_1 := \{\mathbf{H} \in \mathbb{H}^{r\times r} : \mathbf{y}^T\, \mathbf{H}(0)\, \overline{\mathbf{y}} = 0\}$, which was already considered in [21]. Observe that $\mathbf{F} \in V_1$. Using Proposition 3.5 in [21], it follows that the transfer operator T, restricted to V_1 has spectral radius < 1, contradicting (10). Note that Proposition 3.5. in [21] (see also [18])) can be used since the integrability of $\tilde{\mathbf{\Pi}}_n\, \tilde{\mathbf{\Pi}}_n^\star$ is ensured by (7). ■

Remark. 1. Note, that condition (7) can be replaced by a weaker condition in this theorem. We only need to ensure that the solution vector $\mathbf{\Phi}$ is contained in $L^2(\mathbb{R})$.

2. Note that for a given $(r-s)\times r$ matrix $\tilde{\mathbf{M}}(\omega)$ of trigonometric polynomials we can find an $r\times s$ matrix $\mathbf{M}(\omega)$ with $\tilde{\mathbf{M}}(\omega)\,\mathbf{M}(\omega) = \mathbf{0}$, everytimes. Introducing the determinants $\Delta_{d_1,\ldots,d_{r-s}}(\omega)$ of $(r-s)\times(r-s)$ submatrices of $\tilde{\mathbf{M}}(\omega)$ consisting of the d_1th, d_2th,..., and the d_{r-s}th column of $\tilde{\mathbf{M}}(\omega)$, and letting $\mathbf{M}(\omega) = (m_{l,k}(\omega))_{l=1,\ldots,r,k=1,\ldots,s}$, choose

$$m_{l,k}(\omega) = \begin{cases} 0 & l<k \text{ or } l>k+r-s, \\ \Delta_{l+1,\ldots,l+r-s}(\omega) & l=k, \\ (-1)^{r-s}\,\Delta_{l,\ldots,l+r-s-1}(\omega) & l=k+r-s, \\ (-1)^{k+l}\,\Delta_{l,\ldots,k-1,k+1,\ldots,l+r-s}(\omega) & k<l<k+r-s. \end{cases}$$

4. Necessary and sufficient conditions for orthonormality

Let us introduce the following definition. We say that an $\omega\in(0,2\pi)$ is cyclic, if there exists an integer $m\geq 2$ such that $2^m\omega\equiv\omega \pmod{2\pi}$. Equivalently, ω is cyclic, if and only if it is of the form

$$\omega = \frac{2\pi\mu}{2^m-1}$$

for some $m\in\mathbb{N}$, $m\geq 2$ and $\mu\in\{1,\ldots,2^m-1\}$. Considering a cyclic $\omega_1\in(0,\,2\pi)$, we can associate a cycle of numbers $\{\omega_1,\ldots,\omega_m\}$, where $\omega_k := 2\,\omega_{k+1} \pmod{2\pi}$ and $\omega_m := 2\,\omega_1 \pmod{2\pi}$. With $\omega_1 = \frac{2\pi\mu}{2^m-1}$ we obtain

$$\omega_k = \frac{2^{m-k+2}\pi\mu}{2^m-1} \pmod{2\pi} \qquad (k=1,\ldots,m).$$

It can easily be shown that ω and $\omega+\pi \pmod{2\pi}$ can not both be cyclic.

We are now ready to state the following theorem giving the relation between orthonormality conditions in terms of the transfer operator T (see [21]) and direct conditions to the two-scale symbol $\mathbf{P}$ (see [10,22]).

Theorem 6. *Let $\boldsymbol{P}(\omega)$ be of the form (3), satisfying the basic conditions of Proposition 1 and (7). Further, let $\mathbf{a}$ be a suitable right eigenvector of $\boldsymbol{P}(0)$ corresponding to the eigenvalue 1. Then the following assertions are equivalent:*

(A) *The solution vector $\boldsymbol{\Phi}$ of (1) determined by (8) is orthonormal (up to multiplication with a constant).*

(B) *(i) The two-scale symbol $\boldsymbol{P}(\omega)$ does not satisfy the linear dependence condition (9), and*
(ii) for all cycles $\{\omega_1,\ldots,\omega_m\}$ in $(0,\,2\pi)$ for the operation $\omega\to 2\omega \pmod{2\pi}$, and for all $\mathbf{x}\in\mathbb{R}^r$ there exists an $n\in\mathbb{N}_0$ and an $k\in\{0,\ldots,m-1\}$ such that

$$\mathbf{x}^T\,\boldsymbol{Q}_m^n\,\boldsymbol{P}(\omega_1)\ldots\boldsymbol{P}(\omega_k)\,\boldsymbol{P}(\omega_{k+1}+\pi) \neq \boldsymbol{0}^T,$$

where $\boldsymbol{Q}_m := \boldsymbol{P}(\omega_1)\ldots\boldsymbol{P}(\omega_m)$.

(C) *The transfer operator T possesses a simple eigenvalue* 1.

Proof: We show that (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (A).

1. (A) $\Rightarrow$ (B): Let $\mathbf{\Phi}$ be an orthonormal solution vector of (1). We show the necessity of the conditions (B). Here, we only need to consider condition (ii), the condition (i) is already proved to be necessary in Theorem 5.

We show that: If (B) (ii) is not satisfied, then the solution vector $\mathbf{\Phi}$ is not L^2-stable (and hence not orthonormal). Assume that there is a cycle $\{\omega_1, \ldots, \omega_m\}$ and a nonzero vector $\mathbf{x} \in \mathbb{R}^r$ satisfying $\mathbf{x}^T\, \mathbf{Q}_m^n\, \mathbf{P}(\omega_1) \ldots \mathbf{P}(\omega_k)\, \mathbf{P}(\omega_{k+1} + \pi) = \mathbf{0}^T$ for all $n \in \mathbb{N}_0$ and all $k \in \{0, \ldots, m-1\}$. We show that $\mathbf{x}^T\, \mathbf{\Omega}(\omega_m)\, \overline{\mathbf{x}} = 0$:
By definition of a cycle, ω_k equals $2\omega_{k+1} \pmod{2\pi}$, and $\omega_m = 2\omega_1 \pmod{2\pi}$. For an arbitrary, fixed $l \in \mathbb{Z}$, choose l_1, such that $\omega_m + 2\pi l = 2\omega_1 + 2\pi l_1$. Hence, the refinement equation implies that $\mathbf{x}^T\, \hat{\mathbf{\Phi}}(\omega_m + 2\pi l) = \mathbf{x}^T\, \hat{\mathbf{\Phi}}(2\omega_1 + 2\pi l_1)$ is either (for odd l_1) equal to $\mathbf{x}^T\, \mathbf{P}(\omega_1 + \pi)\, \hat{\mathbf{\Phi}}(\omega_1 + \pi l) = 0$ or (for even l_1) to $\mathbf{x}^T\, \mathbf{P}(\omega_1)\, \hat{\mathbf{\Phi}}(\omega_1 + \pi l_1) = \mathbf{x}^T\, \mathbf{P}(\omega_1)\, \hat{\mathbf{\Phi}}(2\omega_2 + 2\pi l_2)$ for some l_2 with $|l_2| \le 1 + |l_1|/2$. Now, we can use the same argument again, and find that $\mathbf{x}^T\, \mathbf{P}(\omega_1)\, \hat{\mathbf{\Phi}}(2\omega_2 + 2\pi l_2)$ either equals to $\mathbf{x}^T\, \mathbf{P}(\omega_1)\, \mathbf{P}(\omega_2 + \pi)\, \hat{\mathbf{\Phi}}(\omega_2 + \pi l_2) = 0$ or to $\mathbf{x}^T\, \mathbf{P}(\omega_1)\, \mathbf{P}(\omega_2)\, \hat{\mathbf{\Phi}}(2\omega_3 + 2\pi l_3)$ and so on. If $\mathbf{x}^T\, \hat{\mathbf{\Phi}}(\omega_m + 2\pi l)$ is of the form $\mathbf{x}^T\, \mathbf{P}(\omega_1) \ldots \mathbf{P}(\omega_m)\, \hat{\mathbf{\Phi}}(\omega_m + \pi l_m) = \mathbf{x}^T\, \mathbf{Q}_m\, \hat{\mathbf{\Phi}}(\omega_m + \pi l_m)$, we can keep going through the cycle as before. The procedure comes to an end, since the sequence $\{|l_k|\}_{k \ge 1}$ is monotonly decreasing; in fact we have $|l_{k+1}| \le 1 + |l_k|/2$.

Hence, it follows that $\mathbf{x}^T\, \hat{\mathbf{\Phi}}(\omega_m + 2\pi l) = 0$ for all $l \in \mathbb{Z}$ implying that

$$\mathbf{x}^T \sum_{l \in \mathbb{Z}} \hat{\mathbf{\Phi}}(\omega_m + 2\pi l)\, \hat{\mathbf{\Phi}}(\omega_m + 2\pi l)^\star\, \overline{\mathbf{x}} = \mathbf{x}^T\, \mathbf{\Omega}(\omega_m)\, \overline{\mathbf{x}} = 0$$

and contradicting the positive definiteness of the autocorrelation symbol $\mathbf{\Omega}(\omega)$.

2. In the next steps we show that (B) $\Rightarrow$ (C).

Let $\mathbf{H}_0 \in \mathbb{H}^{r \times r}$ be an eigenmatrix of T corresponding to the eigenvalue 1. Since, by (7), $\mathbf{I}$ is already an eigenmatrix of T to 1, we have to show that, under the assumptions (B), the assertion $\mathbf{H}_0(\omega) \neq c\,\mathbf{I}$ leads to a contradiction.

So, let us suppose that $\mathbf{H}(\omega) \neq c\,\mathbf{I}$. Observe that $T\,\mathbf{H}_0(\omega) = \mathbf{H}_0(\omega)$ implies that $T\,\mathbf{H}_0(\omega)^\star = \mathbf{H}_0(\omega)^\star$. Since each matrix is representable as a sum of a hermitian and an antihermitian part, we can restrict us to the case that $\mathbf{H}_0(\omega)$ is hermitian. Hence, the eigenvalues of $\mathbf{H}_0(\omega)$ are real. Introducing the minimum and the maximum eigenvalue of $\mathbf{H}_0$:

$$\lambda_{\min} := \min_\omega \lambda_1(\omega) \quad \text{with} \quad \lambda_1(\omega) := \min\{\lambda : \mathbf{H}_0(\omega)\mathbf{x} = \lambda\,\mathbf{x},\, \mathbf{x} \neq 0\},$$
$$\lambda_{\max} := \min_\omega \lambda_2(\omega) \quad \text{with} \quad \lambda_2(\omega) := \max\{\lambda : \mathbf{H}_0(\omega)\mathbf{x} = \lambda\,\mathbf{x},\, \mathbf{x} \neq 0\},$$

we consider the matrices

$$\mathbf{F}_1(\omega) := \mathbf{H}_0(\omega) - \lambda_{\min}\,\mathbf{I}, \qquad \mathbf{F}_2(\omega) := \lambda_{\max}\,\mathbf{I} - \mathbf{H}_0(\omega).$$

Assuming that $\lambda_{\min} = \lambda_1(\omega_0)$, $\lambda_{\max} = \lambda_2(\tilde{\omega}_0)$, it follows that the matrices $\mathbf{F}_1(\omega)$, $\mathbf{F}_2(\omega)$, both are hermitian and positive semidefinite with det $\mathbf{F}_1(\omega_0) = 0$ and det $\mathbf{F}_2(\tilde{\omega}_0) = 0$. Further, observe that both, $\mathbf{F}_1(\omega)$ and $\mathbf{F}_2(\omega)$, are eigenmatrices of T to the eigenvalue 1.

3. We show that there is an $\omega_1 \in (0,\, 2\pi)$ such that either det $\mathbf{F}_1(\omega_1) = 0$ or det $\mathbf{F}_2(\omega_1) = 0$: If this assertion were not true, then we would have det $\mathbf{F}_1(\omega) \neq 0$ and det $\mathbf{F}_2(\omega) \neq 0$ for all $\omega \in (0, 2\pi)$. But the determinants of $\mathbf{F}_1$ and $\mathbf{F}_2$ have at least one zero by construction, hence det $\mathbf{F}_1(0) =$ det $\mathbf{F}_2(0) = 0$. That means, there exist nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x}^T\,\mathbf{F}_1(0)\,\overline{\mathbf{x}} = \mathbf{y}^T\,\mathbf{F}_2(0)\,\overline{\mathbf{y}} = 0$. In case of $\mathbf{x} = c\,\mathbf{y}$, it follows from the definition of $\mathbf{F}_1$ and $\mathbf{F}_2$ that

$$\mathbf{x}^T\,(\mathbf{H}_0(0) - \lambda_{\min}\mathbf{I})\,\overline{\mathbf{x}} = \mathbf{x}^T\,(\lambda_{\max}\mathbf{I} - \mathbf{H}_0(0))\,\overline{\mathbf{x}} = 0,$$

i.e., $\mathbf{x}^T(\lambda_{\max} - \lambda_{\min})\,\mathbf{I}\overline{\mathbf{x}} = 0$, implying $\lambda_{\min} = \lambda_{\max}$. Hence, all eigenvalues of $\mathbf{H}_0(\omega)$ are equal, and $\mathbf{H}_0(\omega) = c\,\mathbf{I}$. This contradicts our assumption.

So, we only need to consider the case that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. Using that $T\,\mathbf{F}_\nu(\omega) = \mathbf{F}_\nu(\omega)$ $(\nu = 1, 2)$, it follows that

$$\begin{aligned} 0 = \mathbf{x}^T\,\mathbf{F}_1(0)\,\overline{\mathbf{x}} &= \mathbf{x}^T\,\mathbf{P}(0)\,\mathbf{F}_1(0)\,\mathbf{P}(0)^\star\overline{\mathbf{x}} + \mathbf{x}^T\,\mathbf{P}(\pi)\,\mathbf{F}_1(\pi)\,\mathbf{P}(\pi)^\star\overline{\mathbf{x}} \\ 0 = \mathbf{y}^T\,\mathbf{F}_2(0)\,\overline{\mathbf{y}} &= \mathbf{y}^T\,\mathbf{P}(0)\,\mathbf{F}_2(0)\,\mathbf{P}(0)^\star\overline{\mathbf{y}} + \mathbf{y}^T\,\mathbf{P}(\pi)\,\mathbf{F}_2(\pi)\,\mathbf{P}(\pi)^\star\overline{\mathbf{y}} \end{aligned}$$

implying that

$$\mathbf{x}^T\,\mathbf{P}(\pi)\,\mathbf{F}_1(\pi)\,\mathbf{P}(\pi)^\star\overline{\mathbf{x}} = \mathbf{y}^T\,\mathbf{P}(\pi)\,\mathbf{F}_2(\pi)\,\mathbf{P}(\pi)^\star\overline{\mathbf{y}} = 0.$$

Since $\mathbf{F}_1(\pi)$, $\mathbf{F}_2(\pi)$ were supposed to be nonsingular, it follows that $\mathbf{x}^T\mathbf{P}(\pi) = \mathbf{y}^T\,\mathbf{P}(\pi) = \mathbf{0}^T$. Hence, (7) implies that $\mathbf{x}^T\mathbf{P}(0)\,\mathbf{P}(0)^\star\,\overline{\mathbf{x}} = \mathbf{x}^T\,\overline{\mathbf{x}}$ and $\mathbf{y}^T\mathbf{P}(0)\,\mathbf{P}(0)^\star\,\overline{\mathbf{y}} = \mathbf{y}\,\overline{\mathbf{y}}$. But this is a contradiction to the basic condition that $\mathbf{P}(0)$ possesses a simple eigenvalue 1.

4. Let $\mathbf{F}$ be one of the matrices $\mathbf{F}_1$, $\mathbf{F}_2$ satisfying det $\mathbf{F}(\omega_1) = 0$ for some $\omega_1 \in (0, 2\pi)$. Let $\mathbf{x}_1$ be a right eigenvector corresponding to the eigenvalue 0, *i.e.*, $\mathbf{x}_1^\star\,\mathbf{F}(\omega_1)\,\mathbf{x}_1 = 0$. Hence,

$$\begin{aligned} 0 &= \mathbf{x}_1^\star\,\mathbf{F}(\omega_1)\,\mathbf{x} = \mathbf{x}_1^\star(T\,\mathbf{F})(\omega_1)\,\mathbf{x}_1 \\ &= \mathbf{x}_1^\star\,\mathbf{P}(\frac{\omega_1}{2})\,\mathbf{F}(\frac{\omega_1}{2})\mathbf{P}(\frac{\omega_1}{2})^\star\mathbf{x}_1 + \mathbf{x}_1^\star\,\mathbf{P}(\frac{\omega_1}{2} + \pi)\,\mathbf{F}(\frac{\omega_1}{2} + \pi)\,\mathbf{P}(\frac{\omega_1}{2} + \pi)^\star\mathbf{x}_1 \end{aligned}$$

implying that

$$\mathbf{x}_1^\star\,\mathbf{P}(\frac{\omega_1}{2})\,\mathbf{F}(\frac{\omega_1}{2})\mathbf{P}(\frac{\omega_1}{2})^\star\mathbf{x}_1 = 0$$

as well as

$$\mathbf{x}_1^\star\,\mathbf{P}(\frac{\omega_1}{2} + \pi)\,\mathbf{F}(\frac{\omega_1}{2} + \pi)\,\mathbf{P}(\frac{\omega_1}{2} + \pi)^\star\mathbf{x}_1 = 0.$$

Observing that, by (7), never both $\mathbf{x}_1^\star\,\mathbf{P}(\frac{\omega_1}{2})$ and $\mathbf{x}_1^\star\,\mathbf{P}(\frac{\omega_1}{2}+\pi)$ can be zero vectors at the same time, it follows that either det $\mathbf{F}(\frac{\omega_1}{2}) = 0$ or det $\mathbf{F}(\frac{\omega_1}{2}+\pi) = 0$ with corresponding left eigenvectors $\mathbf{x}_1^\star\,\mathbf{P}(\frac{\omega_1}{2})$ and $\mathbf{x}_1^\star\,\mathbf{P}(\frac{\omega_1}{2}+\pi)$ to the eigenvalue 0, respectively. Let ω_2 be equal to $\frac{\omega_1}{2}$ or to $\frac{\omega_1}{2}+\pi$ such that det $\mathbf{F}(\omega_2) = 0$. With $\mathbf{x}_2^\star := \mathbf{x}_1^\star\,\mathbf{P}(\omega_2)$ we then have $\mathbf{x}_2^\star\,\mathbf{F}(\omega_2)\,\mathbf{x}_2 = 0$. We again apply the transfer operator and find a further zero of det $\mathbf{F}$, namely either $\frac{\omega_2}{2}$ and $\frac{\omega_2}{2}+\pi$.

Continuing this process, to each $\omega_1 \in (0, 2\pi)$ with $\mathbf{x}_1^\star\,\mathbf{F}(\omega_1)\,\mathbf{x}_1 = 0$ we can find a chain of zeros of det $\mathbf{F}$, $\omega_1, \omega_2, \ldots$ in $(0, 2\pi)$, where ω_{k+1} is either $\frac{\omega_k}{2}$ or $\frac{\omega_k}{2}+\pi$ (or, equivalently, $\omega_k = 2\,\omega_{k+1}\,(\mathrm{mod}\ 2\pi)$) with $\mathbf{x}_k^\star\,\mathbf{F}(\omega_k)\,\mathbf{x}_k = 0$ and $\mathbf{x}_{k+1}^\star := \mathbf{x}_k^\star\,\mathbf{P}(\omega_{k+1})$ $(k = 1, 2, \ldots)$. Since det $\mathbf{F}(\omega)$ is a trigonometric polynomial, it can only have a finite number of different zeros. The case det $\mathbf{F}(\omega) = 0$ for all ω can not happen, since, by Theorem 5 (c), this assertion contradicts the assumption (i) of (B). Hence, the chain $\omega_1, \omega_2, \ldots$ is finite, *i.e.,* there is an $l \in \mathbb{N}$, such that $\omega_l = \omega_k$ for some $k < l$, that means, $\omega_k = 2^{l-k}\omega_l\,(\mathrm{mod}\ 2\pi) = 2^{l-k}\,\omega_k\,(\mathrm{mod}\ 2\pi)$. Thus, ω_k is cyclic, and moreover, by

$$\omega_1 2^{l-k}\,(\mathrm{mod}\ 2\pi) = 2^k\,\omega_k\,2^{l-k}\,(\mathrm{mod}\ 2\pi) = 2^k\,\omega_k\,(\mathrm{mod}\ 2\pi) = \omega_1\,(\mathrm{mod}\ 2\pi),$$

ω_1 is cyclic. The same procedure can be applied to all zeros of det $\mathbf{F}(\omega)$ in $(0,\, 2\pi)$ yielding a finite number of cycles. In particular, det $\mathbf{F}(\omega)$ only has cyclic zeros.

5. Since ω and $\omega + \pi$ can not both be cyclic at the same time, it follows from det $\mathbf{F}(\omega) = 0$ that det $\mathbf{F}(\omega+\pi) \neq 0$. Using again that $\mathbf{F}(\omega) = T\,\mathbf{F}(\omega)$, we find for a cycle $\{\omega_1, \ldots, \omega_m\}$ with $\mathbf{x}_k^\star\,\mathbf{F}(\omega_k)\,\mathbf{x}_k = 0$ and det $\mathbf{F}(\omega_{k+1}+\pi) \neq 0$ from

$$0 = \mathbf{x}_k^\star\,\mathbf{P}(\omega_{k+1})\,\mathbf{F}(\omega_{k+1})\,\mathbf{P}(\omega_{k+1})^\star\,\mathbf{x}_k + \mathbf{x}_k^\star\,\mathbf{P}(\omega_{k+1}+\pi)\,\mathbf{F}(\omega_{k+1}+\pi)\,\mathbf{P}(\omega_{k+1}+\pi)^\star\,\mathbf{x}_k$$

that $\mathbf{x}_k^\star\,\mathbf{P}(\omega_{k+1}+\pi) = \mathbf{0}^T$ $(k = 0, \ldots, m-1)$. This process can be continued by going again through the cycle. Hence, there is a cycle $\{\omega_1, \ldots, \omega_m\}$ and a nonzero vector $\mathbf{x}$, which does not satisfy the condition (B) (ii), and we have found the contradiction.

6. We finally observe that (C) $\Rightarrow$ (A): Let the eigenvalue 1 of T be simple. By (7), I is already an eigenmatrix of T to the eigenvalue 1. Since the autocorrelation symbol $\mathbf{\Omega}(\omega)$ is also an eigenmatrix of T corresponding to 1, it is a multiple of $\mathbf{I}$. Hence, $\mathbf{\Phi}$ is orthonormal (up to multiplication with a constant). ■

Remarks. 1. The above Theorem 6 can be seen as a generalization of Theorem 6.3.5 in [5] showing the equivalence between Lawton's condition [16] and Cohen's criteria [1].
2. As shown in [10], condition (B) (ii) is already satisfied if there is no cycle $\{\omega_1, \ldots, \omega_m\}$ with det $\mathbf{P}(\omega_k + \pi) = 0$ $(k = 0, \ldots, n)$.

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