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Sparse Phase Retrieval of One-Dimensional Signals by Prony's Method

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In this paper, we show that sparse signals f representable as a linear combination of a finite number N of spikes at arbitrary real locations or as a finite linear combination of B-splines of order m with arbitrary real knots can be almost surely recovered from $\mathcal{O}(N^2)$ intensity measurements $\left|\mathcal{F}[f](\omega)\right|^2$ up to trivial ambiguities. The constructive proof consists of two steps, where in the first step Prony's method is applied to recover all parameters of the autocorrelation function and in the second step the parameters of f are derived. Moreover, we present an algorithm to evaluate f from its Fourier intensities and illustrate it at different numerical examples.

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1. INTRODUCTION

Phase retrieval problems occur in many scientific fields, particularly in optics and communications. They have a long history with rich literature regarding uniqueness of solutions and existence of reliable algorithms for signal reconstruction, see e.g., [1] and references therein. Usually, the challenge in solving one-dimensional phase retrieval problems is to overcome the strong ambiguousness by determining appropriate further information on the solution signal. Previous literature on characterization of ambiguities of the phase retrieval problem with given Fourier intensities is often concerned with the discrete problem, where a signal $\mathbf{x}$ in $\mathbb{R}^N$ or $\mathbb{C}^N$ has to be recovered. For an overview on the occurring trivial and non-trivial ambiguities in the discrete setting we refer to our survey [2]. The behavior of the solution set under additional constraints has been studied for instance in Beinert and Plonka [2, 3], Beinert [4, 5].

1.1. Contribution of This Paper

In this paper, we consider the continuous one-dimensional sparse phase retrieval problem to reconstruct a complex-valued signal from the modulus of its Fourier transform. Applications of this problem occur in electron microscopy, wave front sensing, laser optics [6, 7] as well as in X-ray crystallography and speckle imaging [8]. For the posed problem, we will show that for sparse signals the given Fourier intensities are already sufficient for an almost sure unique recovery, and we will give a construction algorithm to recover f.

We assume that the sparse signal is either of the form

$$f(t) = \sum_{j=1}^{N} c_j^{(0)}\, \delta(t - T_j) \tag{1.1}$$

or, for $m > 0$,

$$f(t) = \sum_{j=1}^{N} c_j^{(m)} \, B_{j,m}(t) \tag{1.2}$$

with $c_j^{(m)} \in \mathbb{C}$, $T_j \in \mathbb{R}$ for $j = 1, \dots, N$, where δ denotes the Delta distribution, and $B_{j,m}$ is the B-spline of order m being determined by the (real) knots $T_j < T_{j+1} < \dots < T_{j+m}$. We want to recover these signals from the Fourier intensities $|\widehat{f}(\omega)|^2$ and will show that only $\mathcal{O}(N^2)$ samples are needed to recover f, i.e., all coefficients $c_j^{(m)}$, $j = 1, \dots, N$ and knots T_j, $j = 1, \dots, N + m$, almost surely up to trivial ambiguities. The proposed procedure is constructive and consists in two steps. In a first step, we employ Prony's method to determine the coefficients and frequencies of the exponential sum $\left|\mathcal{F}\left[f\right](\omega)\right|^2$. These frequencies are of the form $T_j - T_k$ with $j, k \in \{1, \dots, N + m\}$. If these knot differences $T_j - T_k$ are pairwise different for $j \neq k$, then we can use this information in a second reconstruction step to compute the knots T_j and the coefficients c_j, and thus the desired signal.

1.2. Related Work on Sparse Phase Retrieval

While the general phase retrieval problem has been extensively studied for a long time, the special case of sparse phase retrieval grew to a strongly emerging field of research only recently, particularly often connected with ideas from compressed sensing. Most of the papers consider a discrete setting, where the N-dimensional real or complex k-sparse vector $\mathbf{x}$ has to be reconstructed from measurements of the more general form $|\langle \mathbf{a}_j, \mathbf{x} \rangle|^2$ with vectors $\mathbf{a}_j$ forming the rows of a measurement matrix $\mathbf{A} \in \mathbb{C}^{M \times N}$. The needed number M of measurements depends on the sparsity k.

If $\mathbf{A}$ presents rows of a Fourier matrix, this setting is close to the sparse phase retrieval problem considered in optics, see e.g., [9]. Here the problem is first rewritten as (non-convex) rank minimization problem, then a tight convex relaxation is applied and the optimization problem is solved by a re-weighted l_1-minimization method. The related approach in Eldar et al. [10] employs the magnitudes of the short-time Fourier transform and applies the occurring redundancy for unique recovery of the desired signal. A corresponding reconstruction algorithm is then based on an adaptation of the GESPAR algorithm in Shechtman et al. [11].

In Li and Voroninski [12], the measurement matrix $\mathbf{A}$ is taken with random rows and the PhaseLift approach [13] leads to a convex optimization problem that recovers the sparse solution with high probability. Employing a thresholded gradient descent algorithm to a non-convex empirical risk minimization problem that is derived from the phase retrieval problem, Cai et al. [14] have established the minimax optimal rates of convergence for noisy sparse phase retrieval under sub-exponential noise.

Other papers rely on the compressed sensing approach to construct special frame vectors $\mathbf{a}_j$ to ensure uniqueness of the phase retrieval problem with high probability, where the number of needed vectors is $\mathcal{O}(k)$, see e.g., [15–17].

We would like to emphasize that all approaches employing general or random measurement matrices in phase retrieval are quite different in nature from our phase retrieval problem based on Fourier intensity measurements. In this paper, we want to stick on considering Fourier intensity measurements because of their particular relevance in practice.

Early attempts to exploit sparsity of a discrete signal for unique recovery using Fourier intensities go back to unpublished manuscripts by Yagle [18, 19], where a variation of Prony's method is applied in a non-iterative algorithm to sparse signal and image reconstruction. Unfortunately, the algorithm proposed there not always determines the signal support correctly.

The continuous one-dimensional phase retrieval problem has been rarely discussed in the literature, see [5, 8, 20–22]. In the preprint [8], the authors also considered the recovery of sparse continuous signals of the form (1.1). However, in that paper the sparse phase retrieval problem is in turn transferred into a turnpike problem that is computationally expensive to solve. Moreover there exist cases, where a unique solution cannot be found, see [23]. Our method circumvents this problem by proposing an iterative procedure to fix the signal support (resp. the knots of the signal represented as a B-spline function) where the corresponding signal coefficients are evaluated simultaneously.

1.3. Organization of This Paper

In Section 2, we shortly recall the mathematical formulation of the considered sparse phase retrieval problem and the notion of trivial ambiguities of the phase retrieval problem that always occur.

Section 3 is devoted to the special case of phase retrieval for signals of the form (1.1). Using Prony's method, we give a constructive proof for the unique recovery of the N-sparse signal f up to trivial ambiguities using $3/2\, N(N-1)+1$ Fourier intensity measurements. Here we have to assume that the knot differences $T_j - T_k$ are pairwise different.

In Section 4, the ansatz is generalized to the unique recovery of spline functions of the form (1.2) where we need to employ $3/2(N + m)(N + m - 1) + 1$ Fourier intensity measurements. In Section 5, we present an explicit algorithm for the considered sparse phase retrieval problem and illustrate it at different examples.

2. TRIVIAL AMBIGUITIES OF THE PHASE RETRIEVAL PROBLEM

We wish to recover an unknown complex-valued signal $f : \mathbb{R} \to \mathbb{C}$ of the form (1.1) or (1.2) with compact support from its Fourier intensity $|\mathcal{F}[f]|$ given by

$$\left|\mathcal{F}\big[f\big](\omega)\right| := \left|\widehat{f}(\omega)\right| := \left|\int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t\right| \quad (\omega \in \mathbb{R}). \tag{2.1}$$

For the spike function in (1.1), we interpret the Fourier integral in (2.1) in a distributional sense, i.e., $\mathcal{F}[\delta(\cdot - T_j)](\omega) = \mathrm{e}^{-\mathrm{i}\omega T_j}$. Unfortunately, the recovery of the signal f is complicated because of the well-known ambiguousness of the phase retrieval problem. Transferring [2, Proposition 2.1] to our setting, we can recover f

only up to the following ambiguities, which immediately follow from the properties of the Fourier transform.

Proposition 2.1. *Let f be a signal of the form (1.1) or a non-uniform spline function of the form (1.2). Then*

(i) *the rotated signal $\mathrm{e}^{\mathrm{i}\alpha} f$ for $\alpha \in \mathbb{R}$,*
(ii) *the time shifted signal $f(\cdot - t_0)$ for $t_0 \in \mathbb{R}$,*
(iii) *and the conjugated and reflected signal $\overline{f(-\cdot)}$*

have the same Fourier intensity $|\mathcal{F}[f]|$.

Although the ambiguities in Proposition 2.1 always occur, they are of minor interest because of their close relation to the original signal. For this reason, we call ambiguities caused by rotation, time shift, conjugation and reflection, or by combinations of these *trivial*. In the following, we will show that for the considered sparse signals the remaining non-trivial ambiguities only occur in rare cases.

3. PHASE RETRIEVAL FOR DISTRIBUTIONS WITH DISCRETE SUPPORT

Initially, we restrict ourselves to the recovery of signals f of the form (1.1) with complex-valued coefficients $c_j^{(0)}$, spike locations $T_1 < \cdots < T_N$, and Fourier transform.

$$\widehat{f}(\omega) = \sum_{j=1}^{N} c_j^{(0)} \, \mathrm{e}^{-\mathrm{i}\omega T_j} \qquad (\omega \in \mathbb{R}).$$

The known squared Fourier intensity $|\mathcal{F}[f]|^2$ can be represented by

$$\left|\widehat{f}(\omega)\right|^2 = \sum_{j=1}^{N}\sum_{k=1}^{N} c_j^{(0)} \overline{c}_k^{(0)} \, \mathrm{e}^{-\mathrm{i}\omega(T_j - T_k)}. \tag{3.1}$$

Thus, in order to recover f being determined by the coefficients $c_j^{(0)} \in \mathbb{C}$ and the knots $T_j \in \mathbb{R}, j = 1, \dots, N$, we will first recover the differences of the frequencies $T_j - T_k$ and the corresponding products of coefficients $c_j^{(0)} \overline{c}_k^{(0)}$ in (3.1) and then derive the desired parameters of f in a second step.

3.1. First Step: Parameter Recovery by Prony's Method

Let us assume that the knot differences $T_j - T_k$ in (3.1) are pairwise different for $j \neq k$. The squared Fourier intensity can then be written in the form

$$\begin{aligned} P(\omega) &:= \left|\widehat{f}(\omega)\right|^2 = \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}\omega\tau_\ell} \\ &= \gamma_0 + \sum_{\ell=1}^{N(N-1)/2} \left(\gamma_\ell \, \mathrm{e}^{-\mathrm{i}\omega\tau_\ell} + \overline{\gamma}_\ell \, \mathrm{e}^{\mathrm{i}\omega\tau_\ell}\right), \end{aligned} \tag{3.2}$$

where we assume that the frequencies τ_ℓ, $\ell = -N(N-1)/2, \dots, N(N-1)/2$ are ordered by size. Obviously, the frequency differences τ_ℓ satisfy $\tau_{-\ell} = -\tau_\ell$. Further, each $\tau_\ell > 0$ corresponds to a difference $T_j - T_k$ for some $j > k$, and the related coefficient γ_ℓ then equals to $c_j^{(0)} \overline{c}_k^{(0)}$. For the zero frequency $\tau_0 = 0$, we have $\gamma_0 := \sum_{j=1}^{N} |c_j^{(0)}|^2$.

In the first step, we want to recover all frequencies τ_ℓ and the corresponding coefficients γ_ℓ, $\ell = 0, \dots, N(N-1)/2$ of $P(\omega)$. However, at this stage, the bijective mapping between $\ell > 0$ and (j, k) with $j > k$ such that $\tau_\ell = T_j - T_k$ will be still unknown and needs to be found in a second reconstruction step. In order to recover the frequency differences τ_ℓ and the unknown coefficients γ_ℓ from the exponential sum (3.2) we employ Prony's method [24, 25].

Let $h > 0$ be chosen such that $h\tau_\ell < \pi$ for all $\ell = 1, \dots, N(N-1)/2$. Using the intensity values $P(hk) = |\mathcal{F}[f](hk)|^2$, $k = 0, \dots, 2N(N-1)+1$, the unknown parameters γ_ℓ and τ_ℓ in (3.2) can be determined by exploiting the algebraic Prony polynomial $\Lambda(z)$ defined by

$$\Lambda(z) := \prod_{\ell=-N(N-1)/2}^{N(N-1)/2} \left(z - \mathrm{e}^{-\mathrm{i}h\tau_\ell}\right) = \sum_{k=0}^{N(N-1)+1} \lambda_k \, z^k, \tag{3.3}$$

where λ_k denote the coefficients in the monomial representation of $\Lambda(z)$. Obviously, $\Lambda(z)$ is always a monic polynomial, which means that $\lambda_{N(N-1)+1} = 1$.

Using the definition of the Prony polynomial $\Lambda(z)$ in (3.3), we observe that

$$\begin{aligned} \sum_{k=0}^{N(N-1)+1} \lambda_k \, P\big(h(k+m)\big) &= \sum_{k=0}^{N(N-1)+1} \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \lambda_k \gamma_\ell \, \mathrm{e}^{-\mathrm{i}h(k+m)\tau_\ell} \\ &= \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}hm\tau_\ell} \, \Lambda\left(\mathrm{e}^{-\mathrm{i}h\tau_\ell}\right) = 0 \end{aligned}$$

for $m = 0, \dots, N(N-1)$. Consequently, the vector of remaining coefficients $\boldsymbol{\lambda} := (\lambda_0, \dots, \lambda_{N(N-1)})^{\mathrm{T}}$ of the Prony polynomial $\Lambda(z)$ can be determined by solving the system of linear equations

$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{h} \tag{3.4}$$

with $\boldsymbol{H} := \big(P(h(k+m))\big)_{m,k=0}^{N(N-1)}$ and $\boldsymbol{h} := \big(P(h(N(N-1)+1+m))\big)_{m=0}^{N(N-1)}$. Since the Hankel matrix $\boldsymbol{H}$ can be written as

$$\boldsymbol{H} = \boldsymbol{V}^{\mathrm{T}} \operatorname{diag}\big(\gamma_{-N(N-1)/2}, \dots, \gamma_{N(N-1)/2}\big) \, \boldsymbol{V}$$

with the Vandermonde matrix $\boldsymbol{V} := \left(\mathrm{e}^{-hk\tau_\ell}\right)_{\ell=-N(N-1)/2, k=0}^{N(N-1)/2, N(N-1)+1}$, the system of linear equations (3.4) possesses a unique solution if and only if the unimodular values $\mathrm{e}^{-\mathrm{i}h\tau_\ell}$ differ pairwise for $\ell = -N(N-1)/2, \dots, N(N-1)/2$. This assumption has been ensured by choosing an h such that $h\tau_\ell \in (-\pi, \pi)$, since the τ_ℓ had been supposed to be pairwise different.

Knowing the coefficients λ_k of $\Lambda(z)$, we can determine the unknown frequencies τ_ℓ by evaluating the roots of the Prony polynomial (3.3). The coefficients γ_ℓ can now be computed by solving the over-determined equation system

$$\sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}hk\tau_\ell} = P\big(hk\big) \qquad (k = 0, \dots, 2N(N-1)+1) \tag{3.5}$$

with a Vandermonde-type system matrix.

The procedure summarized above is Prony's method, adapted to the non-negative exponential sum $P(\omega)$ in (3.2). In the numerical experiments in Section 5, we will apply the approximate Prony method (APM) in Potts and Tasche [26]. APM is based on the above considerations but it is numerically more stable and exploits the special properties $\gamma_{-\ell} = \overline{\gamma}_\ell$ and $\tau_{-\ell} = -\tau_\ell$ for $\ell = 0, \dots, N(N-1)/2$.

Let us now investigate the question, how many intensity values are at least necessary for the recovery of $P(\omega)$ in (3.2). Counting the number of unknowns of $P(\omega)$ in (3.2), we only need to recover the $3/2\, N(N-1)+1$ real values γ_0 and $\operatorname{Re}\gamma_\ell$, $\operatorname{Im}\gamma_\ell$, τ_ℓ, for $\ell = 1, \dots N(N-1)/2$. We will show now that using the special structure of the real polynomial $P(\omega)$ in (3.2) and of the Prony polynomial $\Lambda(z)$ in (3.3), we indeed need only $3/2\, N(N-1)+1$ exact equidistant real measurements $P(kh)$, $k = 0, \dots, 3/2\, N(N-1)$ to recover all parameters determining $P(\omega)$. This can be seen as follows.

Reconsidering $\Lambda(z)$ in (3.3) with $\tau_0 = 0$ and $\tau_\ell = -\tau_{-\ell}$, we obtain

$$\begin{aligned}\Lambda(z) &= (z-1) \prod_{\ell=1}^{N(N-1)/2} \Big(z - \mathrm{e}^{\mathrm{i}h\tau_\ell}\Big)\Big(z - \mathrm{e}^{-\mathrm{i}h\tau_\ell}\Big) \\ &= (z-1) \prod_{\ell=1}^{N(N-1)/2} \big(z^2 - 2z\cos(h\tau_\ell) + 1\big) = \sum_{k=0}^{N(N-1)+1} \lambda_k \, z^k,\end{aligned}$$

where all occurring coefficients λ_k are real. Moreover, since

$$z^{-(N(N-1)+1)/2} \Lambda(z) = (z^{1/2} - z^{-1/2}) \prod_{\ell=1}^{N(N-1)/2} \big(z - 2\cos(h\tau_\ell) + z^{-1}\big)$$

is antisymmetric, it follows that

$$\lambda_{N(N-1)+1-k} = -\lambda_k \qquad (k = 0, \dots, N(N-1)/2),$$

and particularly $\lambda_{N(N-1)+1} = -\lambda_0 = 1$. In order to determine the unknown coefficients λ_k, $k = 1, \dots, N(N-1)/2$ of

$$\Lambda(z) = \sum_{k=0}^{N(N-1)/2} \lambda_k \Big(z^k - z^{N(N-1)+1-k}\Big),$$

we employ (3.2) and observe that for $m = 0, \dots, N(N-1)/2 - 1$,

$$\begin{aligned}&\sum_{k=0}^{N(N-1)/2} \lambda_k \big[P(h(k+m)) - P(h(N(N-1)+1+m-k))\big] \\ &= \sum_{k=0}^{N(N-1)/2} \lambda_k \Bigg[\sum_{\ell=1}^{N(N-1)/2} \gamma_\ell \Big(\mathrm{e}^{-\mathrm{i}h(k+m)\tau_\ell} - \mathrm{e}^{-\mathrm{i}h(N(N-1)+1+m-k)\tau_\ell}\Big) \\ &\quad + \sum_{\ell=1}^{N(N-1)/2} \overline{\gamma}_\ell \Big(\mathrm{e}^{\mathrm{i}h(k+m)\tau_\ell} - \mathrm{e}^{\mathrm{i}h(N(N-1)+1+m-k)\tau_\ell}\Big)\Bigg] \\ &= \sum_{\ell=1}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}hm\tau_\ell} \sum_{k=0}^{N(N-1)/2} \lambda_k \Big(\mathrm{e}^{-\mathrm{i}hk\tau_\ell} - \mathrm{e}^{-\mathrm{i}h(N(N-1)+1-k)\tau_\ell}\Big) \\ &\quad + \sum_{\ell=1}^{N(N-1)/2} \overline{\gamma}_\ell \, \mathrm{e}^{\mathrm{i}hm\tau_\ell} \sum_{k=0}^{N(N-1)/2} \lambda_k \Big(\mathrm{e}^{\mathrm{i}hk\tau_\ell} - \mathrm{e}^{\mathrm{i}h(N(N-1)+1-k)\tau_\ell}\Big) \\ &= \sum_{\ell=1}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}hm\tau_\ell} \Lambda(\mathrm{e}^{-\mathrm{i}h\tau_\ell}) + \sum_{\ell=1}^{N(N-1)/2} \overline{\gamma}_\ell \, \mathrm{e}^{\mathrm{i}hm\tau_\ell} \lambda(\mathrm{e}^{\mathrm{i}h\tau_\ell}) = 0.\end{aligned}$$

Therefore, the vector of unknown coefficients $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_{N(N-1)/2})^{\mathrm{T}}$ can be evaluated from the system

$$\begin{aligned}&\sum_{k=1}^{N(N-1)/2} \lambda_k \big[P(h(k+m)) - P(h(N(N-1)+1+m-k))\big] \\ &= \big[P(hm) - P(h(N(N-1)+1+m))\big] \\ &\quad (m = 0, \dots, N(N-1)/2 - 1).\end{aligned}$$

The frequency differences τ_ℓ are then extracted from the zeros of $\Lambda(z)$, and the coefficients γ_ℓ, $\ell = 0, \dots, N(N-1)/2$, are computed as in (3.5) but with $k = 0, \dots, 3/2\, N(N-1)$.

3.2. Second Step: Unique Signal Recovery

Having determined the frequency differences τ_ℓ as well as the corresponding coefficients γ_ℓ of (3.2), we want to reconstruct the parameters T_j and $c_j^{(0)}$, $j = 1, \dots, N$, of f in (1.1) in a second step.

Theorem 3.1. *Let f be a signal of the form (1.1), whose knot differences $T_j - T_k$ differ pairwise for $j, k \in \{1, \dots, N\}$ with $j \neq k$, and whose coefficients satisfy $|c_1^{(0)}| \neq |c_N^{(0)}|$. Further, let h be a step size such that $h(T_j - T_k) \in (-\pi, \pi)$ for all j, k. Then f can be uniquely recovered from its Fourier intensities $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 3/2\, N(N-1)$ up to trivial ambiguities.*

Proof. Applying Prony's method to the given data $|\mathcal{F}[f](h\ell)|$, we can compute the frequency differences τ_ℓ and the related coefficients γ_ℓ of the squared Fourier intensity (3.2). We denote by $\mathcal{T} := \{\tau_\ell\}_{\ell=1}^{N(N-1)/2}$ the list of obtained positive frequencies ordered by size. Now, we need to recover the mapping $\ell \to (j,k)$ such that $\tau_\ell = T_j - T_k$, where we can assume that $j > k$ for $\ell > 0$.

Obviously, the maximal distance $\tau_{N(N-1)/2}$ is now equal to the length $T_N - T_1$ of the unknown f in (1.1). Due to the trivial shift ambiguity, we can assume without loss of generality that $T_1 = 0$ and $T_N = \tau_{N(N-1)/2}$. Further, the second largest distance $\tau_{(N(N-1)/2)-1}$ corresponds either to $T_{N-1} - T_1$ or to $T_N - T_2$. Due to the trivial reflection and conjugation ambiguity, we can assume that $T_{N-1} = T_{N-1} - T_1 = \tau_{(N(N-1)/2)-1}$. By definition, there exists a $\tau_{\ell^*} > 0$ in our sequence of parameters $\mathcal{T}$ such that

$\tau_{\ell^*} + \tau_{(N(N-1)/2)-1} = \tau_{N(N-1)/2}$, and τ_{ℓ^*} is hence equal to the knot difference $T_N - T_{N-1}$. Thus, we obtain

$$c_N^{(0)}\bar{c}_1^{(0)} = \gamma_{N(N-1)/2}, \qquad c_{N-1}^{(0)}\bar{c}_1^{(0)} = \gamma_{(N(N-1)/2)-1}, \quad \text{and} \quad c_N^{(0)}\bar{c}_{N-1}^{(0)} = \gamma_{\ell^*}.$$

These equations lead us to

$$c_N^{(0)} = \frac{\gamma_{N(N-1)/2}}{\bar{c}_1^{(0)}}, \qquad c_{N-1}^{(0)} = \frac{\gamma_{(N(N-1)/2)-1}}{\bar{c}_1^{(0)}},$$

and thus to

$$\left|c_1^{(0)}\right|^2 = \frac{\gamma_{N(N-1)/2}\overline{\gamma_{(N(N-1)/2)-1}}}{\gamma_{\ell^*}}.$$

Since f can only be recovered up to a global rotation, we can assume that $c_1^{(0)}$ is real and non-negative, which allows us to determine the coefficients $c_1^{(0)}$, $c_N^{(0)}$, and $c_{N-1}^{(0)}$ in a unique way.

Having fixed $T_N = T_N - T_1 = \tau_{N(N-1)/2}$ and $T_{N-1} = T_{N-1} - T_1 = \tau_{(N(N-1)/2)-1}$ we notice that the third largest distance $\tau_{(N(N-1)/2)-2}$ is either equal to $T_N - T_2$ or to $T_{N-2} - T_1 = T_{N-2}$. As before, there exists a frequency τ_{ℓ^*} such that $\tau_{(N(N-1)/2)-2} + \tau_{\ell^*} = \tau_{N(N-1)/2}$.

Case 1: If $\tau_{(N(N-1)/2)-2} = T_N - T_2$, then we have

$$\begin{aligned}\tau_{\ell^*} &= \tau_{N(N-1)/2} - \tau_{(N(N-1)/2)-2} = (T_N - T_1) - (T_N - T_2)\\ &= T_2 - T_1\end{aligned}$$

with the related coefficient $\gamma_{\ell^*} = c_2^{(0)}\bar{c}_1^{(0)}$. Moreover, we have $\gamma_{(N(N-1)/2)-2} = c_N^{(0)}\bar{c}_2^{(0)}$ such that

$$c_2^{(0)} = \frac{\gamma_{\ell^*}}{\bar{c}_1^{(0)}} = \frac{\overline{\gamma}_{(N(N-1)/2)-2}}{\bar{c}_N^{(0)}}. \tag{3.6}$$

Case 2: If $\tau_{(N(N-1)/2)-2} = T_{N-2} - T_1$, then we have

$$\begin{aligned}\tau_{\ell^*} &= \tau_{N(N-1)/2} - \tau_{(N(N-1)/2)-2} = (T_N - T_1) - (T_{N-2} - T_1)\\ &= T_N - T_{N-2}\end{aligned}$$

with the related coefficient $\gamma_{\ell^*} = c_N^{(0)}\bar{c}_{N-2}^{(0)}$ and $\gamma_{(N(N-1)/2)-2} = c_{N-2}^{(0)}\bar{c}_1^{(0)}$. Thus,

$$c_{N-2}^{(0)} = \frac{\overline{\gamma}_{\ell^*}}{\bar{c}_N^{(0)}} = \frac{\gamma_{(N(N-1)/2)-2}}{\bar{c}_1^{(0)}}. \tag{3.7}$$

However, only one of the two equalities in (3.6) and (3.7) can be true, since if both were true then $\gamma_{\ell^*}\bar{c}_N^{(0)} = \bar{c}_1^{(0)}\overline{\gamma}_{(N(N-1)/2)-2}$ and $\bar{c}_1^{(0)}\overline{\gamma}_{\ell^*} = \bar{c}_N^{(0)}\gamma_{(N(N-1)/2)-2}$ lead to

$$\left|\frac{c_N^{(0)}}{c_1^{(0)}}\right| = \left|\frac{\gamma_{(N(N-1)/2)-2}}{\gamma_{\ell^*}}\right| = \left|\frac{c_1^{(0)}}{c_N^{(0)}}\right|$$

contradicting the assumption that $|c_N^0| \neq |c_1^0|$. Consequently, either the equation in (3.6) or the equation in (3.7) holds true and we can either determine T_2 with $c_2^{(0)}$ or T_{N-2} with $c_{N-2}^{(0)}$. Removing all frequency differences τ_ℓ from the sequence of distances $\mathcal{T}$ that correspond to the differences $T_j - T_k$ of the recovered knots, we can repeat this approach to find the remaining coefficients and knots of f inductively. □

If we identify the space of complex-valued signals of the form (1.1) with the real space $\mathbb{R}^{3N}$, the condition that two knot differences $T_{j_1} - T_{k_1}$ and $T_{j_2} - T_{k_2}$ are equal for fixed indices j_1, j_2, k_1, and k_2 defines a hyperplane with Lebesgue measure zero. An analogous observation follows for the condition $|c_1^{(0)}| = |c_N^{(0)}|$. The signals excluded in Theorem 3.1 hence form a negligible null set.

Corollary 3.2. *Almost all signals f in (1.1) can be uniquely recovered from their Fourier intensities $\left|\mathcal{F}[f]\right|$ up to trivial ambiguities.*

Remark 3.3. 1. Since the proof of Theorem 3.1 is constructive, it can be used to recover an unknown signal (1.1) analytically and numerically. If the number N of spikes is known beforehand then the assumption of Theorem 3.1 can be simply checked during the computation. If the assumption regarding pairwise different distances $T_j - T_k$ is not satisfied, then the application of Prony's method in the first step yields less than $N(N-1)+1$ pairwise distinct frequency differences τ_ℓ. The second assumption $|c_N^0| \neq |c_1^0|$ can be verified in the second step, where $c_1^{(0)}$, $c_{N-1}^{(0)}$, and $c_N^{(0)}$ are evaluated.

2. A similar phase retrieval problem had been transferred to a turnpike problem in Ranieri et al. [8]. The turnpike problem deals with the recovery of the knots T_j from an unlabeled set of distances. Although this problem is solvable under certain conditions, a backtracing algorithm can have exponential complexity in the worst case, see [27].

4. RETRIEVAL OF SPLINE FUNCTIONS WITH ARBITRARY KNOTS

In this section, we generalize our findings to spline functions of order $m \geq 1$. Let us recall that the B-splines $B_{j,m}$ in (1.2) being generated by the knot sequence $T_1 < \cdots < T_{N+m}$ are recursively defined by

$$B_{j,m}(t) := \frac{t-T_j}{T_{j+m-1}-T_j}\, B_{j,m-1}(t) + \frac{T_{j+m}-t}{T_{j+m}-T_{j+1}}\, B_{j+1,m-1}(t)$$

with

$$B_{j,1}(t) := \mathbb{1}_{[T_j,T_{j+1})}(t) := \begin{cases}1 & t \in [T_j, T_{j+1}),\\ 0 & \text{else},\end{cases}$$

see for instance [28, p. 131]. Further, we notice that for $0 \leq k \leq m-2$ the kth derivative of the spline f in (1.2) is given by

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) = \sum_{j=1}^{N+k} c_j^{(m-k)}\, B_{j,m-k}(t), \tag{4.1}$$

where the coefficients $c_j^{(m-k)}$ are recursively defined by

$$c_j^{(m-k)} := \left(m-k\right)\,\frac{c_j^{(m-k+1)} - c_{j-1}^{(m-k+1)}}{T_{j+m-k} - T_j} \qquad (j = 1, \dots, N+k),$$

with the convention that $c_0^{(m-k+1)} = c_{N+k}^{(m-k+1)} = 0$, see [28, p. 139]. For $k = m-1$, Equation (4.1) coincides with a step function, i.e., with the right derivative of the linear spline $f^{(m-2)}$. Further, in a distributional manner, the mth derivative of f is given by

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) = \sum_{j=1}^{N+m} c_j^{(0)}\,\delta(t - T_j) \tag{4.2}$$

with the coefficients

$$c_1^{(0)} := c_1^{(1)}, \quad c_{N+m}^{(0)} := -c_{N+m-1}^{(1)}, \quad c_j^{(0)} := c_j^{(1)} - c_{j-1}^{(1)} \quad (j = 2, \dots, N+m-1),$$

and the Dirac delta distribution δ.

Applying the Fourier transform to (4.2), we now obtain

$$\widehat{f^{(m)}}(\omega) = (\mathrm{i}\omega)^m \hat{f}(\omega) = \sum_{j=1}^{N+m} c_j^{(0)} \mathrm{e}^{-\mathrm{i}\omega T_j}. \tag{4.3}$$

and thus

$$\omega^{2m}\left|\hat{f}(\omega)\right|^2 = \sum_{j=1}^{N+m}\sum_{k=1}^{N+m} c_j^{(0)}\bar{c}_k^{(0)}\,\mathrm{e}^{-\mathrm{i}\omega(T_j - T_k)}. \tag{4.4}$$

Since the exponential sum on the right-hand side of (4.4) has exactly the same structure as the exponential sum in (3.2), we can immediately generalize Theorem 3.1 by considering

$$P(\omega) := \omega^{2m}\left|\hat{f}(\omega)\right|^2 = \sum_{\ell=-(N+m)(N+m-1)/2}^{(N+m)(N+m-1)/2} \gamma_\ell\,\mathrm{e}^{-\mathrm{i}\omega\tau_\ell}. \tag{4.5}$$

Theorem 4.1. *Let f be a spline function of the form (1.2) of order m, whose knot distances $T_j - T_k$ differ pairwise for $j, k \in \{1, \dots, N+m\}$ with $j \neq k$, and whose coefficients satisfy $|c_1^{(0)}| \neq |c_{N+m}^{(0)}|$. Further, let h be a step size such that $h(T_j - T_k) \in (-\pi, \pi)$ for all j, k. Then f can be uniquely recovered from its Fourier intensities $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 3/2(N+m)(N+m-1)$ up to trivial ambiguities.*

Proof. The statement can be established by proceeding in the same manner as in Section 3. First we apply Prony's method to the given samples $(h\ell)^{2m}|\mathcal{F}[f](h\ell)|^2$ with $\ell = 0, \dots, 3/2(N+m)(N+m-1)$ in order to determine the coefficients and frequencies of $P(\omega)$ in (4.5). In a second step, the values $c_j^{(0)}$ and T_j in (4.3) can be determined analytically as discussed in Theorem 3.1. Reversing the definition of $c_j^{(m-k)}$, we can finally compute the unknown coefficients $c_j^{(m)}$ by

$$c_j^{(1)} = c_j^{(0)} + c_{j-1}^{(1)} \qquad (j = 1, \dots, N+m-1)$$

and

$$c_j^{(m-k+1)} = \tfrac{T_{j+m-k} - T_j}{m-k}\,c_j^{(m-k)} + c_{j-1}^{(m-k+1)} \qquad (j = 1, \dots, N+k-1)$$

with $c_0^{(1)} := 0$ and $c_0^{(m-k+1)} := 0$, which finishes the proof. □

Corollary 4.2. *Almost all spline functions f of order m in (1.2) can be uniquely recovered from their Fourier intensities $\left|\mathcal{F}[f]\right|$ up to trivial ambiguities.*

5. NUMERICAL EXPERIMENTS

Since the proofs of Theorem 3.1 and Theorem 4.1 are constructive, they can be straightforwardly transferred to numerical algorithms to recover a spline function from its Fourier intensity. However, Prony's classical method introduced in subsection 3.1 is numerically unstable with respect to inexact measurements and to frequencies lying close together. For this reason, there are numerous approaches to improve the classical method. In order to verify Theorem 3.1 and Theorem 4.1 numerically, we apply the so-called approximate Prony method (APM) proposed by Potts and Tasche [26, Algorithm 4.7] for recovery of parameters of an exponential sum of the form

$$P(\omega) = \sum_{\ell=-M}^{M} \gamma_\ell\,\mathrm{e}^{-\mathrm{i}\omega\tau_\ell} \tag{5.1}$$

with $\tau_\ell = -\tau_{-\ell}$ and $\gamma_\ell = \overline{\gamma}_{-\ell}$. The algorithm can be summarized as follows, where the exact number $2M+1$ of the occurring frequencies in (5.1) needs not be known beforehand.

Algorithm 5.1 Approximate Prony method [26]

Input: upper bound $L \in \mathbb{N}$ of the number $2M+1$ of exponentials; measurements $P(hk)$ with $k = 0, \dots, 2\breve{M}$ and $\breve{M} \geq L$; accuracies ε_1, ε_2, and ε_3.

1. Compute a right singular vector $\boldsymbol{\lambda}^{(1)} := (\lambda_k^{(1)})_{k=0}^{L}$ corresponding to the smallest singular value of the rectangular Hankel matrix $\boldsymbol{H} := (P(h(k+m)))_{k,m=0}^{2N-L,L}$.
2. Evaluate the roots $z_j^{(1)} = r_j^{(1)}\,\mathrm{e}^{\mathrm{i}\omega_j^{(1)}}$ of the polynomial $\Lambda^{(1)}(z) := \sum_{k=0}^{L}\lambda_k^{(1)} z^k$ with $\omega_j^{(1)} \in [0, \pi)$ and $|r_j^{(1)} - 1| \leq \varepsilon_1$.
3. Compute a right singular vector $\boldsymbol{\lambda}^{(2)} := (\lambda_k^{(2)})_{k=0}^{L}$ corresponding to the second smallest singular value of the rectangular Hankel matrix $\boldsymbol{H} := (P(h(k+m)))_{k,m=0}^{2N-L,L}$.
4. Evaluate the roots $z_j^{(2)} = r_j^{(2)}\,\mathrm{e}^{\mathrm{i}\omega_j^{(2)}}$ of the polynomial $\Lambda^{(2)}(z) := \sum_{k=0}^{L}\lambda_k^{(2)} z^k$ with $\omega_j^{(2)} \in [0, \pi)$ and $|r_j^{(2)} - 1| \leq \varepsilon_1$.
5. Determine all frequencies of the form $\omega_\ell := 1/2\,(\omega_j^{(1)} + \omega_k^{(2)})$ if there exist indices j and k with $|\omega_j^{(1)} - \omega_k^{(2)}| \leq \varepsilon_2$, and denote the number of found frequencies by $\widetilde{M}$.

6. Determine the coefficients $c_j^{(m)}$ by solving the over-determined equation system (5.2).

Output: knots T_j and coefficients $c_j^{(m)}$ of the signal (1.1) ($m = 0$) or the spline function in (1.2) ($m > 0$).

Example 5.1. In the first numerical example, we consider a spike function as in (1.1) with 15 spikes. More precisely, the locations T_j and the coefficients $c_j^{(0)}$ of the true spike function f are given in **Table 1**. In order to recover f from the Fourier intensity measurements $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \ldots, 1000$, we apply Algorithm 2 with the accuracies $\varepsilon := 10^{-3}$, $\varepsilon_1 := 10^{-5}$, $\varepsilon_2 := 10^{-7}$, and $\varepsilon_3 := 10^{-10}$. In order to ensure that $h(T_j - T_k) \in (-\pi, \pi)$ as assumed in Theorem 3.1, we chose $h := {}^{0.95\,(T_{15}-T_1)}/{}_\pi$. The results of the phase retrieval algorithm and the absolute errors of the knots and coefficients of the recovered spike function are shown in **Figure 1**. Although the approximate Prony method has to recover 211 knot differences, the knots and coefficients of f are reconstructed very accurately. ○

Example 5.2. In the second example, we consider the piecewise quadratic spline function f of order $m = 3$ as in (1.2) with $N = 7$ coefficients and $N + m = 10$ knots as given in **Table 2**. To recover f from its Fourier intensity measurements $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \ldots, 400$ and with $h = {}^{0.95\,(T_{10}-T_1)}/{}_\pi$, we again apply Algorithm 2. As accuracies for the phase retrieval algorithm and the approximate Prony method, we choose $\varepsilon := 10^{-3}$, $\varepsilon_1 := 10^{-5}$, $\varepsilon_2 := 10^{-10}$, and $\varepsilon_3 := 10^{-10}$. In **Figure 2**, the recovered function is compared with the true signal. Again, the reconstructed knots and coefficients have only very small absolute errors. ○

6. SUMMARY AND DISCUSSION

In this paper, we have presented a novel approach to recover a sparse continuous-time signal f from finitely many samples of its Fourier intensity $|\mathcal{F}[f]|$.

While the general phase retrieval problem is highly ambiguous, the assumed sparsity of the unknown signal

TABLE 1 | Knots T_j and coefficients $c_j^{(0)}$ of the spike function in Example 5.1.

j	T_j	$c_j^{(0)}$	j	T_j	$c_j^{(0)}$	j	T_j	$c_j^{(0)}$
1	−53.5895	4.910 + 0.000i	6	−28.1475	0.278 + 0.598i	11	1.3755	2.887 + 3.828i
2	−50.2765	−0.165 + 0.814i	7	−22.6005	−1.450 + 3.246i	12	20.0945	−1.423 + 0.397i
3	−49.3765	−2.368 − 1.314i	8	−19.6495	0.508 + 0.243i	13	33.4525	0.023 − 2.039i
4	−42.6915	−0.293 + 0.541i	9	−6.1705	0.073 − 0.528i	14	34.8415	−2.997 + 3.767i
5	−28.3915	−1.841 + 2.589i	10	−3.8985	3.135 + 0.339i	15	53.5895	−0.064 − 0.368i

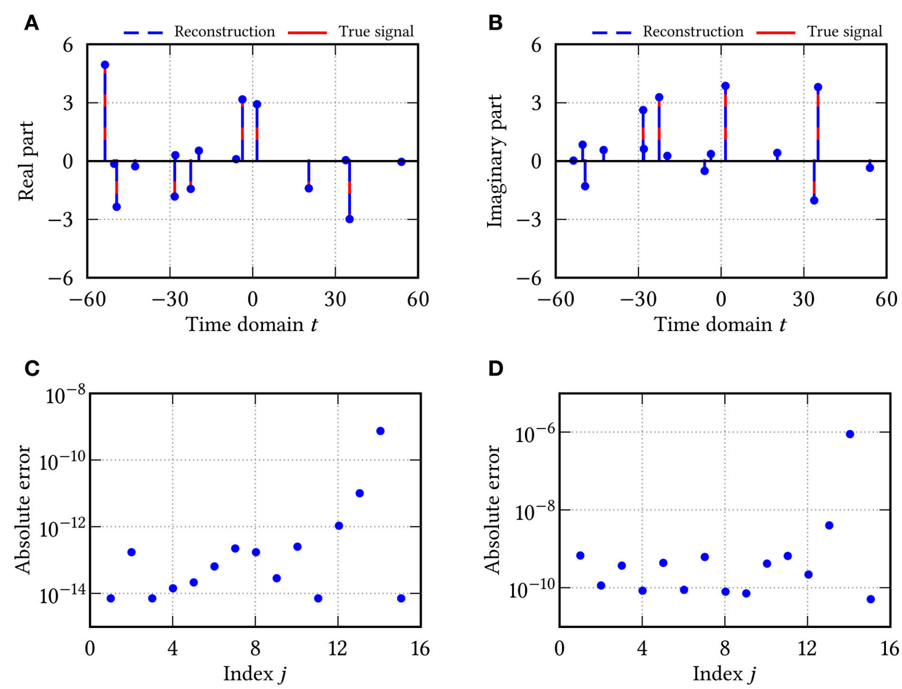

FIGURE 1 | Results of Algorithm 5.2 for the spike function in Example 5.1. (A) Real part of the recovered and true spike function. **(B)** Imaginary part of the recovered and true spike function. **(C)** Absolute error of the recovered knots. **(D)** Absolute error of the recovered coefficients.

TABLE 2 | Knots T_j and coefficients $c_j^{(3)}$ of the spline function in Example 5.2.

j	T_j	$c_j^{(3)}$	j	T_j	$c_j^{(3)}$	j	T_j	$c_j^{(3)}$
1	−17.022	5.342 + 0.000i	5	−7.745	3.597 − 0.334i	8	2.309	—
2	−13.921	−3.569 + 0.132i	6	−4.313	0.554 − 2.251i	9	9.318	—
3	−9.536	0.440 − 1.413i	7	−0.336	−4.072 + 1.433i	10	17.022	—
4	−8.301	−4.685 − 0.499i						

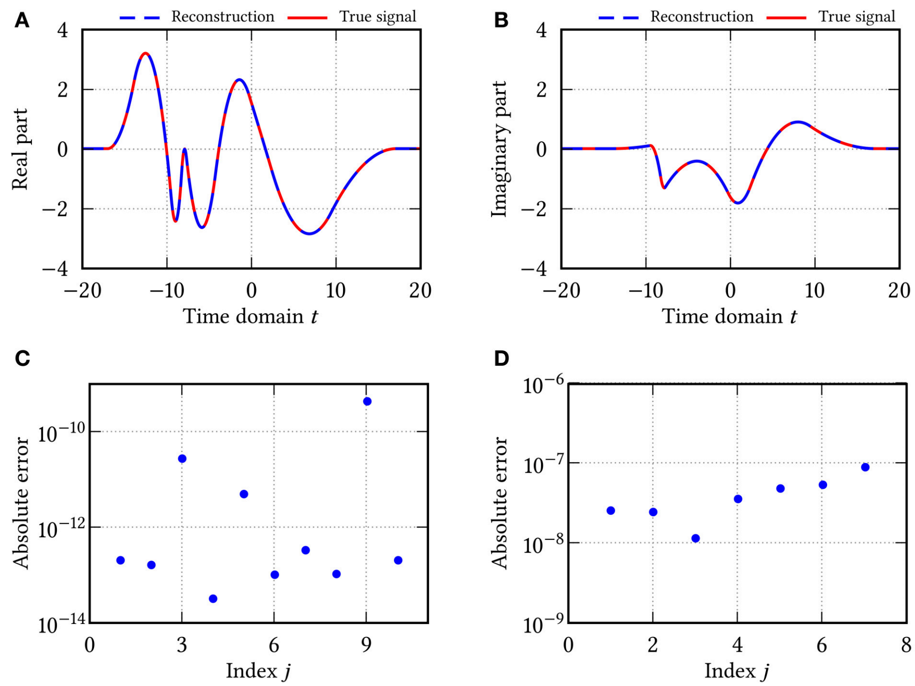

FIGURE 2 | Results of Algorithm 5.2 for the spline function in Example 5.2. (A) Real part of the recovered and true spline function. **(B)** Imaginary part of the recovered and true spline function. **(C)** Absolute error of the recovered knots. **(D)** Absolute error of the recovered coefficients.

surmounts this problem and guarantees uniqueness of the phase retrieval problem up to trivial ambiguities. In many applications, the sparsity assumption arises in a natural manner. For instance, the positions of stars in astronomy [29] or the positions of atoms in a molecule in crystallography [30] correspond to a sparse spike functions.

Here, we have assumed that f is a finite linear combination of spikes or B-splines with arbitrary knots. The new approach consists of two steps, where we have applied Prony's method in a first step to determine the knot differences from the exponential sum $|\mathcal{F}[f]|^2$. Based on this information, we have derived a method to recover the unknown knots and coefficients of f step by step. The significant benefit over the previous approach in Ranieri et al. [8] is the exploitation of the coefficients of $|\mathcal{F}[f]|^2$, which allows the simultaneous recovery of the knots and coefficients of the true signal f always with polynomial complexity. Our method works for all signals whose knot differences are pairwise distinct. Therefore, almost every structured function of the form (1.1) or (1.2) can be uniquely recovered from its Fourier intensity up to trivial ambiguities. In the numerical examples, we show that our methods behaves well in the noise-free setting. Our work is a first step to phase retrieval of spline functions and raises several theoretical and numerical questions.

The considered phase retrieval problem employs Fourier transform intensities. For spike functions, the proposed method can be easily extended to measurements from a canonical linear transform like the Fresnel or the fractional Fourier transform, since these transforms merely correspond to a non-linear modulation of the coefficients, cf. [31]. The phase retrieval problem of spline functions in Section 4 is essentially based on formula (4.3) on the representation of function derivatives in Fourier domain. This property does not generally hold for canonical linear transforms.

The sensitivity of our reconstruction algorithm with respect to noisy measurements depends on the approximate Prony method. In fact, the desired frequency differences possess a very special structure and have to satisfy certain side conditions. For example, the sum of two frequency differences $T_j - T_k$ and $T_\ell - T_j$ is again a frequency difference $T_\ell - T_k$. For strongly disturbed measurements, the recovered frequency differences obtained by the approximate Prony method may not satisfy this special structure, and the second reconstruction step of our method cannot be applied directly. Therefore, it would be interesting to study, how the approximate Prony method can be modified by incorporating the additional structure information.

AUTHOR CONTRIBUTIONS

All authors listed, have made substantial, direct and intellectual contribution to the work, and approved it for publication.

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