Factorization of refinement masks of function vectors

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Dedicated to Prof. E. W. Cheney on the occasion
of his 65th birthday

Abstract. Considering the set of closed shift–invariant subspaces V_j ($j \in \mathbb{Z}$) of $L^2(\mathbb{R})$ generated by a refinable function vector $\mathbf{\Phi}$, we give necessary and sufficient conditions for the refinement mask of $\mathbf{\Phi}$ ensuring controlled approximation order m. In particular, algebraic polynomials can be exactly reproduced in V_0 if and only if the refinement mask of $\mathbf{\Phi}$ can be factorized. The results are illustrated by B–splines with multiple knots.

§1 Introduction

The idea of considering a ladder of imbedded subspaces V_j of a Hilbert space for approximating functions has extensively been used in many applications. In the case of multiresolution analysis of $L^2(\mathbb{R})$, the subspaces V_j are usually *generated* by a single function $\phi \in L^2(\mathbb{R})$,

$$V_j := \mathrm{clos}_{L^2} \, \mathrm{span} \, \{\phi(2^j \cdot -l) : \, l \in \mathbb{Z}\}.$$

In order to ensure the condition $V_j \subset V_{j+1}$ ($j \in \mathbb{Z}$), we need a *refinable* scaling function, *i.e.*, ϕ has to satisfy a functional equation of the type

$$\phi = \sum_{l \in \mathbf{Z}} p_l \, \phi(2 \cdot -l) \quad (\{p_l\}_{l \in \mathbf{Z}} \in l^2). \tag{1.1}$$

A lot of papers have been delt with solutions of (1.1) and with their properties. Functions satisfying (1.1) not only arise in the context of multiresolution analysis and wavelets. They also play an important role in subdivision schemes. By convenient choice of the *refinement mask*

$$P := \sum_{l \in \mathbf{Z}} p_l \, e^{-il \cdot}$$

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one is able to influence properties of ϕ like regularity or support and, at the same time, properties of the generated set of subspaces V_j of $L^2(\mathbb{R})$. The close connection between the refinement mask of ϕ and the structure of the shift–invariant subspaces generated by ϕ is the clue for many successful applications of the theory of multiresolution and corresponding wavelets. For example, V_0 provides controlled approximation order m if and only if the refinement mask P factorizes

$$P(u) = \left(\frac{1+e^{-iu}}{2}\right)^m S(u) \tag{1.2}$$

with an appropriate chosen 2π-periodic function S (cf. [2–4]).
In the last time, also the generalized *multiresolution analysis of multiplicity* r ($r \in \mathbb{N}$) of $L^2(\mathbb{R})$ has been considered in more detail (cf. [5–8]). Now the set of imbedded closed subspaces V_j of $L^2(\mathbb{R})$ is *generated* by a function vector $\mathbf{\Phi} := (\phi_\nu)_{\nu=0}^{r-1}$ with $\phi_\nu \in L^2(\mathbb{R})$ ($\nu = 0, \ldots, r-1$),

$$V_j := \mathrm{clos}_{L^2} \operatorname{span} \{\phi_\nu(2^j \cdot -l) : l \in \mathbf{Z},\, \nu = 0, \ldots, r-1\}.$$

The vector of scaling functions $\mathbf{\Phi}$ has to be *refinable, i.e.*, $\mathbf{\Phi}$ has to satisfy a functional equation of the type

$$\mathbf{\Phi} = \sum_{l \in \mathbf{Z}} \mathbf{P}_l \, \mathbf{\Phi}(2 \cdot -l) \quad (\mathbf{P}_l \in \mathbb{R}^{r \times r}),$$

where the sequences of entries of coefficient matrices $\mathbf{P}_l$ ($l \in \mathbf{Z}$) are in l^2.
The purpose of this paper is an investigation of the structure of the *refinement mask*

$$\mathbf{P} := \sum_{l \in \mathbf{Z}} \mathbf{P}_l \, e^{-il\cdot}, \tag{1.3}$$

if the corresponding shift–invariant subspace V_0 is assumed to provide controlled approximation order m. It turns out that, under some mild conditions on $\mathbf{\Phi}$, algebraic polynomials of degree $< m$ can be exactly reproduced in V_0 if and only if $\mathbf{P}$ can be factorized in a certain manner. This result is a natural generalization of the result (1.2) for a single scaling function.
In Section 2, we will introduce some notations and present the main theorem on the factorization of the refinement mask. The example of B–splines with multiple knots is used to illustrate this result in Section 3.

§2 Factorization of the refinement mask

Let us introduce some notations. Consider the Hilbert space $L^2 = L^2(\mathbb{R})$ of all square integrable functions on $\mathbb{R}$. The Fourier transform of $f \in L^2(\mathbb{R})$

is defined by $\hat{f} := \int_{-\infty}^{\infty} f(x) e^{-ix\cdot}\,\mathrm{d}x$. Further, let $BV(\mathbb{R})$ be the set of all functions which are of bounded variation over $\mathbb{R}$ and normalized by

$$f(-\infty) = f(-\infty+0) = 0, \qquad f(\infty) = f(\infty-0) = 0,$$
$$f(x) = \frac{1}{2}(f(x+0) + f(x-0)) \quad (-\infty < x < \infty).$$

If $f \in L^2(\mathbb{R}) \cap BV(\mathbb{R})$, then the Poisson Summation Formula

$$\sum_{l\in\mathbf{Z}} f(l)\, e^{-iul} = \sum_{j\in\mathbf{Z}} \hat{f}(u + 2\pi j) \quad (u \in \mathbb{R})$$

is satisfied. For a measurable function f on $\mathbb{R}$ and $m \in \mathbb{N}$ let

$$\|f\|_2 := (\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x)^{1/2},$$

$$|f|_{m,2} := \|\mathrm{D}^m f\|_2, \qquad \|f\|_{m,2} := \sum_{k=0}^{m} \|\mathrm{D}^k f\|_2.$$

Here and in the following, D denotes the differential operator $\mathrm{D} := \mathrm{d}/\mathrm{d}\,\cdot$. Let $W_2^m(\mathbb{R})$ be the usual Sobolev space with the norm $\|\cdot\|_{m,2}$. The l^2–norm of a sequence $\mathbf{c} := \{c_l\}_{l\in\mathbf{Z}}$ is defined by $\|\mathbf{c}\|_{l^2} := (\sum_{l\in\mathbf{Z}} |c_l|^2)^{1/2}$.
The set $\mathcal{B}(\mathbf{\Phi}) := \{\phi_\nu(\cdot - l) : l \in \mathbf{Z},\ \nu = 0,\ldots,r-1\}$ forms a *Riesz basis* of V_0 if there exist constants $0 < A \le B < \infty$ with

$$A \sum_{\nu=0}^{r-1} \|\mathbf{c}_\nu\|_{l^2}^2 \le \|\sum_{\nu=0}^{r-1} \sum_{l\in\mathbf{Z}} c_{\nu,l}\, \phi_\nu(\cdot - l)\|_{L^2}^2 \le B \sum_{\nu=0}^{r-1} \|\mathbf{c}_\nu\|_{l^2}^2$$

for any sequences $\mathbf{c}_\nu = \{c_{\nu,l}\}_{l\in\mathbf{Z}}$ $(\nu = 0,\ldots,r-1)$.
We say that V_0, generated by the function vector $\mathbf{\Phi}$, provides *controlled approximation order* m if for each $f \in W_2^m(\mathbb{R})$ there are sequences $\mathbf{c}_\nu^h = \{c_{\nu,l}^h\}_{l\in\mathbf{Z}}$ $(\nu = 0,\ldots,r-1; h > 0)$ such that for a constant c independent of h the following three conditions are satisfied:

(1) $$\|f - h^{-1/2} \sum_{\nu=0}^{r-1} \sum_{l\in\mathbf{Z}} c_{\nu,l}^h\, \phi_\nu(\cdot/h - l)\|_2 \le c\, h^m\, |f|_{m,2}.$$

(2) We have
$$\|\mathbf{c}_\nu^h\|_{l^2} \le c\, \|f\|_2 \quad (\nu = 0,\ldots,r-1).$$

(3) There is a constant δ independent of h such that for $l \in \mathbf{Z}$

$$\mathrm{dist}\,(lh, \mathrm{supp}\, f) > \delta \quad \Rightarrow \quad c_{\nu,l}^h = 0 \quad (\nu = 0,\ldots,r-1).$$

Now we want to generalize the known result (1.2) for the principal space V_0 and consider the connections between the refinement mask (1.3) and the structure of the finitely generated closed subspace V_0 of $L^2(\mathbb{R})$. First, let us define the $(r \times r)$–matrix $\mathbf{C} := (C_{j,k})_{j,k=0}^{r-1}$ by the vector $\mathbf{y} = (y_0, \ldots, y_{r-1})^{\mathrm{T}} \in \mathbb{R}^r$, $\mathbf{y} \neq \mathbf{o}$. Here and in the following $\mathbf{o}$ denotes the zero vector. Let $j_0 := \min\{j;\, y_j \neq 0\}$ and $j_1 := \max\{j;\, y_j \neq 0\}$. Further, let for all j with $y_j \neq 0$ be $d_j := \min\{k > j, y_k \neq 0\}$. Then for $j_0 < j_1$, the entries of $\mathbf{C}$ are defined for $j, k = 0, \ldots, r-1$ by

$$C_{j,k}(u) := \begin{cases} y_j^{-1} & y_j \neq 0 \quad \text{and} \quad j = k, \\ 1 & y_j = 0 \quad \text{and} \quad j = k, \\ -y_j^{-1} & y_j \neq 0 \quad \text{and} \quad d_j = k, \\ -e^{-iu}/y_{j_1} & j = j_1 \quad \text{and} \quad k = j_0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

For $j_0 = j_1$, $\mathbf{C}$ is a diagonal matrix of the form

$$\mathbf{C}(u) = \operatorname{diag}(\underbrace{1, \ldots, 1}_{j_0}, (1 - e^{-iu})/y_{j_0}, \underbrace{1, \ldots, 1}_{r-1-j_0}). \tag{2.2}$$

In particular, for $r = 1$ we have $\mathbf{C}(u) := (1 - e^{-iu})/y_0$. Now we can show:

Theorem 2.1 *Let $m \in \mathbb{N}$ be fixed, and let $\mathbf{\Phi} := (\phi_\nu)_{\nu=0}^{r-1}$ be a refinable vector of functions $\phi_\nu \in L^2(\mathbb{R}) \cap BV(\mathbb{R})$ satisfying the decay properties $|\phi_\nu(x)| = O(|x|^{-m-1-\epsilon})$ $(x \to \infty)$ for $\nu = 0, \ldots, r-1$ and $\epsilon > 0$. Further, let $\mathcal{B}(\mathbf{\Phi})$ form a Riesz basis of V_0. Then the following assertions are equivalent:*
(a) *The finitely generated subspace V_0 provides controlled approximation order m.*
(b) *Algebraic polynomials of degree $< m$ can be exactly reproduced in V_0.*
(c) *The refinement mask $\boldsymbol{P}$ of $\mathbf{\Phi}$ satisfies the following conditions: The elements of $\boldsymbol{P}$ are $(m-1)$–times continuously differentiable functions in $L^2_{2\pi}(\mathbb{R})$, and there are vectors $\boldsymbol{y}_k \in \mathbb{R}^r$; $\boldsymbol{y}_0 \neq \boldsymbol{o}$ $(k = 0, \ldots, m-1)$ such that for $n = 0, \ldots, m-1$ we have*

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^{\mathrm{T}} (2i)^{k-n} (\mathrm{D}^{n-k}\boldsymbol{P})(0) &= 2^{-n} (\boldsymbol{y}_n)^{\mathrm{T}}, \\ \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^{\mathrm{T}} (2i)^{k-n} (\mathrm{D}^{n-k}\boldsymbol{P})(\pi) &= \boldsymbol{o}^{\mathrm{T}}. \end{aligned} \tag{2.3}$$

(d) *There are vectors $\mathbf{x}_0, \ldots, \mathbf{x}_{m-1} \in \mathbb{R}^r$ such that $\boldsymbol{P}$ factorizes*

$$\boldsymbol{P}(u) = \frac{1}{2^m} \boldsymbol{C}_{m-1}(2u) \ldots \boldsymbol{C}_0(2u)\, \boldsymbol{S}(u)\, \boldsymbol{C}_0(u)^{-1} \ldots \boldsymbol{C}_{m-1}(u)^{-1}, \tag{2.4}$$

where the $(r \times r)$–matrices $\boldsymbol{C}_k$ are defined by $\mathbf{x}_k$ $(k = 0, \dots, m-1)$ as in (2.1)–(2.2) *and $\boldsymbol{S}(u)$ is an $(r \times r)$–matrix with $(m-1)$–times continuously differentiable entries in $L^2_{2\pi}(\mathbb{R})$. In particular, for the determinant of $\boldsymbol{P}$ we have*

$$\det \boldsymbol{P}(u) = \left(\frac{1+e^{-iu}}{2^r}\right)^m \det \boldsymbol{S}(u).$$

In order to prove Theorem 2.1, we extensively use the assertion that a shift–invariant closed finitely generated subspace of $L^2(\mathbb{R})$ provides controlled approximation order m if and only if the generating function vector $\boldsymbol{\Phi}$ satisfies the Strang–Fix conditions of order m, i.e., there is a finitely supported sequence of vectors $\{\mathbf{a}_l\}_{l\in\mathbf{Z}}$ such that $f := \sum_{l\in\mathbf{Z}} \mathbf{a}_l^{\mathrm{T}} \boldsymbol{\Phi}(\cdot - l)$ satisfies

$$\hat{f}(0) \neq 0; \qquad \mathrm{D}^n \hat{f}(2\pi l) = 0 \quad (l \in \mathbf{Z} \setminus \{0\};\ n = 0, \dots, m-1).$$

This equivalence is already shown in [9]. For a complete proof of Theorem 2.1 we refer to [11,12].

Note that the known results for the principle space V_0 (cf. [2–4]) are obtained from Theorem 2.1 in the special case $r = 1$. In particular, (2.3) simplifies to

$$\mathrm{D}^n P(\pi) = 0 \quad (n = 0, \dots, m-1),$$
$$P(0) = 1.$$

Further, for $r = 1$, it follows from (d) that the refinement mask is of the form (1.2) with $S(0) = 1$, where S is a 2π–periodic $(m-1)$–times continuously differentiable function.

It can be shown that the coefficient vectors $\mathbf{y}_k \in \mathbb{R}^r$ $(k = 0, \dots, m-1)$ occuring in Theorem 2.1 (c) satisfy the equalities

$$\sum_{l\in\mathbf{Z}} \left(\sum_{k=0}^{n} \binom{n}{k} l^{n-k} \mathbf{y}_k^{\mathrm{T}} \right) \boldsymbol{\Phi}(x - l) = x^n \quad (x \in \mathbb{R};\ n = 0, \dots, m-1) \tag{2.5}$$

(cf. [11], Theorem 3.2).

§3 Example: B–splines with multiple knots

We are going to illustrate the results of Theorem 2.1. Let $r \in \mathbb{N}$ and $m \in \mathbb{N}_0$ be given integers. We consider equidistant knots with multiplicity r, $x_l := \lfloor l/r \rfloor$ $(l \in \mathbf{Z})$, where $\lfloor x \rfloor$ means the integer part of $x \in \mathbb{R}$. Let $N_\nu^{m,r}$ $(\nu \in \mathbf{Z})$ denote the cardinal B–splines of order m and defect r with respect to the knots $x_\nu, \dots, x_{\nu+m}$. For $x_{\nu+m} > x_\nu$, we have the recursion formulas

$$(x_{\nu+m} - x_\nu) N_\nu^{m,r}(x) = (x - x_\nu)\, N_\nu^{m-1,r}(x) + (x_{\nu+m} - x)\, N_{\nu+1}^{m-1,r}(x)$$

and

$$(x_{\nu+m} - x_\nu)\,(\mathrm{D}N_\nu^{m,r}) = m\,(N_\nu^{m-1,r} - N_{\nu+1}^{m-1,r}). \tag{3.1}$$

For $x_\nu = x_{\nu+m} = 0$, we define $N_\nu^{m,r}$ according to the distribution theory by

$$N_\nu^{m,r} := \frac{\mathrm{D}^{r-m-1-\nu}\delta}{r-1-\nu},$$

where δ denotes the Dirac distribution. Further, note that

$$N_{\nu+lr}^{m,r} = N_\nu^{m,r}(\cdot - l) \qquad (l \in \mathbf{Z}).$$

For $m \geq r$, we have $N_\nu^{m,r} \in C^{m-r-1}(\mathbb{R})$. We put $\mathbf{N}_m^r := (N_\nu^{m,r})_{\nu=0}^{r-1}$ and $\hat{\mathbf{N}}_m^r := (\hat{N}_\nu^{m,r})_{\nu=0}^{r-1}$. In particular, we obtain

$$\hat{\mathbf{N}}_0^r(u) := \left(\frac{(iu)^{r-1}}{r-1}, \ldots, \frac{(iu)^1}{1}, 1\right)^{\mathrm{T}} \qquad (u \in \mathbb{R}).$$

The spline functions $N_\nu^{m,r}(\cdot - l)\,(l \in \mathbf{Z};\, \nu = 0, \ldots r-1)$ form a Riesz basis of the spline space

$$V_0 := \operatorname{clos}_{L_2} \operatorname{span}\{N_\nu^{m,r}(\cdot - l) :\, l \in \mathbf{Z},\, \nu = 0, \ldots, r-1\}$$

(cf. [1]). It is well–known that V_0 provides controlled approximation order m. We want to compute the vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ $(k = 0, \ldots, r-1)$ occuring in Theorem 2.1. First we observe the following recursion relation:

Lemma 3.1 *Let $r \in \mathbb{N}$ be fixed. Then we have for $m \geq 1$*

$$(iu)\,\hat{\boldsymbol{N}}_m^r(u) = m\,\boldsymbol{C}_{m-1}(u)\,\hat{\boldsymbol{N}}_{m-1}^r(u) \quad (u \in \mathbb{R}) \tag{3.2}$$

with $\boldsymbol{C}_{m-1}$ defined by the vector of spline knots

$$\mathbf{x}_{m-1} := (x_m, \ldots, x_{m+r-1})^{\mathrm{T}}$$

as in (2.1)–(2.2).

Proof: Applying Fourier transform to formula (3.1), we find for $u \in \mathbb{R}$

$$(iu)\,\hat{N}_\nu^{m,r}(u) = \frac{m}{x_{\nu+m} - x_\nu}\,(\hat{N}_\nu^{m-1,r}(u) - \hat{N}_{\nu+1}^{m-1,r}(u)).$$

Thus, by $\hat{N}_r^{m-1,r}(u) = e^{-iu}\,\hat{N}_0^{m-1,r}$ the assertion follows for $m > r$. For $m = r$, the B–splines $N_\nu^{m,m}$ $(\nu = 0, \ldots, m-1)$ coincide with the Bernstein polynomials of degree $m-1$ satisfying

$$\mathrm{D}N_\nu^{m,m} = m(N_\nu^{m-1,m} - N_{\nu+1}^{m-1,m}) \quad (\nu = 0, \ldots, m-1)$$

with $N_0^{m-1,m} := \delta/m$, $N_m^{m-1,m} := \delta(\cdot - 1)/m$ and $N_\nu^{m-1,m} := N_{\nu-1}^{m-1,m-1}$ $(\nu = 1, \ldots, m-1)$. Hence, by Fourier transform, the assertion is true for $m = r$. Finally, for $m < r$, the proof follows analogously, observing that

$$\hat{\mathbf{N}}_m^r(u) = \left(\frac{(iu)^{r-m-1}}{r-1}, \ldots, \frac{(iu)^0}{m}, \hat{\mathbf{N}}_m^m(u)\right)^{\mathrm{T}}. \quad \blacksquare$$

Now the following recursion for the refinement mask $\mathbf{P}_m$ of $\mathbf{N}_m^r$ can be shown.

Theorem 3.2 *Let $r \in \mathbb{N}$ be fixed. Then for $m \geq 1$, the refinement mask $\boldsymbol{P}_m$ of $\boldsymbol{N}_m^r$ satisfies the recursion formula*

$$\boldsymbol{P}_m(u) = \frac{1}{2}\,\boldsymbol{C}_{m-1}(2u)\,\boldsymbol{P}_{m-1}(u)\,\boldsymbol{C}_{m-1}(u)^{-1}$$

with $\boldsymbol{C}_{m-1}$ defined by $\mathbf{x}_{m-1} := (x_m, \ldots, x_{m+r-1})^{\mathrm{T}}$ as in (2.1)–(2.2) *and*

$$\boldsymbol{P}_0(u) := \operatorname{diag}(2^{r-1}, \ldots, 2^0).$$

The proof of Theorem 3.2 is similar to that of Theorem 3.1 in [10]. Repeated application of Theorem 3.2 yields

$$\mathbf{P}_m(u) = \frac{1}{2^m}\,\mathbf{C}_{m-1}(2u)\ldots\mathbf{C}_0(2u)\,\mathbf{P}_0(u)\,\mathbf{C}_0(u)^{-1}\ldots\mathbf{C}_{m-1}(u)^{-1}$$

with $\mathbf{C}_k$ defined by the vector of spline knots $\mathbf{x}_k := (x_{k+1}, \ldots, x_{k+r})^{\mathrm{T}}$ $(k = 0, \ldots, m-1)$. Hence, the refinement mask factorizes in the form (2.4) with $\mathbf{S}(u) := \mathbf{P}_0(u)$.

For the computation of the coefficient vectors $\mathbf{y}_k$ $(k = 0, \ldots, m-1)$ we use the relation (2.5). Introducing the polynomial vector

$$\mathbf{Q}(u) := \left(x_{m+\nu} \prod_{\mu=1}^{m-1} (u - x_{\mu+\nu}) \right)_{\nu=0}^{r-1},$$

we have by [13], Theorem 4.21,

$$x^n = \sum_{l \in \mathbf{Z}} (\mathbf{a}_l^n)^{\mathrm{T}}\, \mathbf{N}_m^r(x - l) \quad (n = 0, \ldots, m-1)$$

with

$$\begin{aligned} \mathbf{a}_l^n &= (-1)^n \frac{n!}{(m-1)!}\,(\mathrm{D}^{m-n-1}\mathbf{Q})(-l) \\ &= \sum_{k=0}^{n} \binom{n}{k} l^{n-k}\,(-1)^k \frac{k!}{(m-1)!}\,(\mathrm{D}^{m-k-1}\mathbf{Q})(0). \end{aligned}$$

Hence, for the coefficient vectors in Theorem 2.1 (c) it follows that

$$\mathbf{y}_k := \mathbf{a}_0^k = (-1)^k \frac{k!}{(m-1)!}\,(\mathrm{D}^{m-k-1}\mathbf{Q})(0) \quad (k = 0, \ldots, m-1).$$

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