ON AN ORTHOGONAL BIVARIATE TRIGONOMETRIC SCHAUDER BASIS FOR THE SPACE OF CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper we construct an orthogonal trigonometric Schauder basis in the space $C(\mathbb{T}^2)$ which has a small growth of the polynomial degree. The construction is mainly based on an anisotropic PMRA. The polynomial degree is considered in terms of l_1- and l_∞-norm.

1. INTRODUCTION

In 1914 Faber [3] showed that no polynomial set $\{t_k \colon k \in \mathbb{N}\}$ with $\deg t_k \leq \frac{k}{2}$ can be a basis for the space $C(\mathbb{T})$ of continuous 2π-periodic functions with the uniform norm. Thus, for a long time it remained an open question for which minimal growth of degree the polynomial set $\{t_k \colon k \in \mathbb{N}\}$ is a Schauder basis in the space $C(\mathbb{T})$. Many papers were devoted to this problem producing Schauder bases with smaller and smaller growth of the polynomial degree. An exhaustive review of the history of this problem could be found in the paper of Ul'yanov [13].

Definitive results on this problem were obtained by Privalov [9, 10]. On the one hand, he showed that for any polynomial basis $\{t_k \colon k \in \mathbb{N}\}$ in the space $C(\mathbb{T})$ there exists an $\varepsilon > 0$ such that for sufficiently large k one has $\deg t_k \geq (1+\varepsilon)\frac{k}{2}$. On the other hand, for any such $\varepsilon > 0$ he constructed a trigonometric Schauder basis satisfying $\deg t_k \leq (1+\varepsilon)\frac{k}{2}$.

At the same time, the following question was investigated: How does the additional condition for the basis to be orthogonal affect the growth of the degree? In particular, in [11] Privalov constructed an orthogonal trigonometric Schauder basis $\{t_k \colon k \in \mathbb{N}\}$ in the space $C(\mathbb{T})$ with $\deg t_k \leq \frac{8}{3}k$. Further, this result was improved by Wojtaszczyk and Woźniakowski [14] and later by Offind and Oskolkov [7]. Finally, Lorentz and Sahakian showed in [6] that the additional condition for the basis to be orthogonal does not affect the growth of the degree, i.e. for any $\varepsilon > 0$ they constructed an orthogonal trigonometric Schauder basis satisfying $\deg t_k \leq (1+\varepsilon)\frac{k}{2}$. Their proof was based on Meyer wavelets and corresponding wavelet packets on the real line which then were periodized. In [8], by the help of de la Vallée Poussin means and related polynomial wavelets, a similar basis was constructed with optimal growth of the degree. This construction by means of periodic wavelet and wavelet packet spaces yields an asymptotically optimal estimation of the norm of the corresponding partial sum operator.

The main purpose of this paper is to solve the above-mentioned problem for the functions of two variables, i.e. to construct an orthogonal trigonometric Schauder basis in the space $C(\mathbb{T}^2)$ with as small as possible growth of the polynomial degree. To achieve this, we use ideas of [8] in combination with ideas of an anisotropic PMRA which were recently developed in [5] and [1].

The paper is organized as follows: In Section 2 we first give the necessary definitions and notations and after that we formulate and prove our main result. Section 3 is devoted to the estimation of the norm of the orthogonal projection operator which is essential in the proof of the main result. In this section we prove several auxiliary lemmas to estimate the corresponding norm of the orthogonal projection operator.

In the paper we use the standard notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$ and $\mathbb{C}$ which correspond to the sets of natural, integer, nonnegative integer, real and complex numbers, respectively. By $X^2 = \{\boldsymbol{x} = (x_1, x_2)\colon x_1 \in X,\ x_2 \in X\}$, we denote the Cartesian product of two sets X where X is one of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$ or $\mathbb{C}$. For a finite set $A \subset \mathbb{R}^2$, the notation $|A|$ denotes the number of points of this set. Moreover, $\mathbb{Z}^{2\times 2}$ is a set of all integer matrices of order 2.

2. Orthogonal trigonometric Schauder basis

2.1. **Function spaces.** Let $\mathbb{T}^2 \cong [-\pi, \pi)^2$ be the 2-dimensional torus. As usual, the normed space $L_p(\mathbb{T}^2)$, $1 \leq p < \infty$, consists of all measurable complex-valued functions f that are 2π-periodic in each variable and

$$\|f\|_p = \left(\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} |f(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x}\right)^{1/p}.$$

By $C(\mathbb{T}^2)$, we denote the space of all 2π-periodic in each variable and continuous on $\mathbb{R}^2$ complex-valued functions f equipped with the norm

$$\|f\|_\infty = \sup_{\boldsymbol{x}\in\mathbb{T}^2} |f(\boldsymbol{x})|.$$

The normed space $C(\mathbb{R}^2)$ consists of complex-valued functions f that are continuous on $\mathbb{R}^2$ and

$$\|f\|_\infty = \sup_{\boldsymbol{x}\in\mathbb{R}^2} |f(\boldsymbol{x})|.$$

For arbitrary vectors $\boldsymbol{r} \in \mathbb{Z}_+^2$, we also define spaces

$$C^{\boldsymbol{r}}(\mathbb{T}^2) = \left\{f\colon \frac{\partial f^{r_1+r_2}}{\partial x_1^{r_1} x_2^{r_2}} \in C(\mathbb{T}^2)\right\}$$

and

$$C^{\boldsymbol{r}}(\mathbb{R}^2) = \left\{f\colon \frac{\partial f^{r_1+r_2}}{\partial x_1^{r_1} x_2^{r_2}} \in C(\mathbb{R}^2)\right\}.$$

2.2. **Anisotropic PMRA.** For an arbitrary regular matrix $\mathbf{M} \in \mathbb{Z}^{2\times 2}$, we define a pattern $P(\mathbf{M}) = \mathbf{M}^{-1}\mathbb{Z}^2 \cap \left[-\frac{1}{2}, \frac{1}{2}\right)^2$ and a generating set $G(\mathbf{M}) = \mathbf{M}P(\mathbf{M})$ (see Figure 1). Using a geometrical argument [2, Lemma II.7], one can see that

$$|P(\mathbf{M})| = |P(\mathbf{M}^{\mathrm{T}})| = |G(\mathbf{M})| = |G(\mathbf{M}^{\mathrm{T}})| = |\det \mathbf{M}|.$$

For any function $f \in L_2(\mathbb{T}^2)$, we introduce the shift operator $\mathrm{T}_{\boldsymbol{y}}f = f(\cdot - 2\pi\boldsymbol{y})$, $\boldsymbol{y} \in \mathbb{R}^2$.

Definition 2.1. For a sequence $\{\mathbf{J}_l\}_{l\in\mathbb{N}}$ of regular matrices $\mathbf{J}_l \in \mathbb{Z}^{2\times 2}$, $|\det \mathbf{J}_l| > 1$, and a sequence of spaces $\{V_j\}_{j\in\mathbb{Z}_+}$, $V_j \subset L_2(\mathbb{T}^2)$, we denote $\mathbf{M}_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and

$$\mathbf{M}_j = \mathbf{J}_j \cdot \ldots \cdot \mathbf{J}_1 \mathbf{M}_0, \quad j \in \mathbb{N}.$$

An anisotropic periodic multiresolution analysis in $L_2(\mathbb{T}^2)$ (anisotropic PMRA) is given by the tuple $\left(\{\mathbf{J}_l\}_{l\in\mathbb{N}}, \{V_j\}_{j\in\mathbb{Z}_+}\right)$ if the following properties are fulfilled:

(MR1) For all $j \in \mathbb{Z}_+$ there exists a function $\varphi_j \in V_j$ such that the shifts $\mathrm{T}_{\boldsymbol{y}}\varphi_j$, $\boldsymbol{y} \in P(\mathbf{M}_j)$, constitute a basis for V_j;

(MR2) For all $j \in \mathbb{Z}_+$ it holds $V_j \subset V_{j+1}$;

(MR3) The union of all V_j is dense in $L_2(\mathbb{T}^2)$.
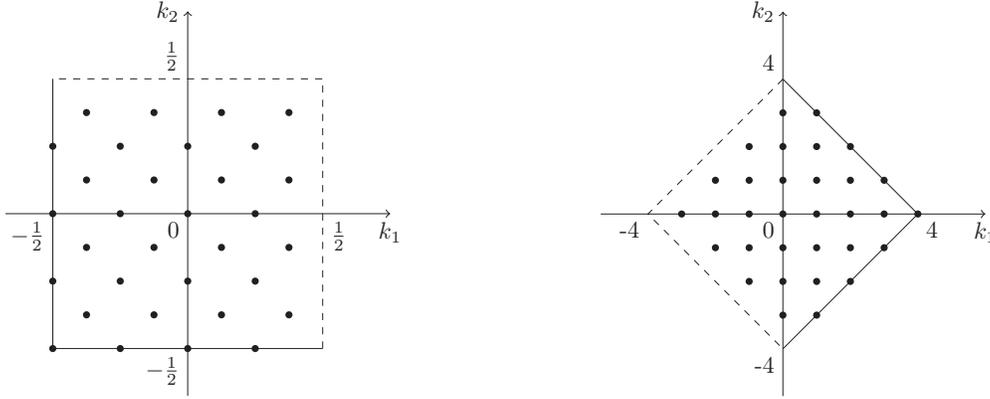
FIGURE 1. The pattern $P(\mathbf{M})$ (left) and the generating set $G(\mathbf{M}^{\mathrm{T}})$ (right) where $\mathbf{M}= \left(\begin{smallmatrix} -4 & 4 \\ -4 & -4 \end{smallmatrix}\right)$.

The space V_j from Definition 2.1 is called $\mathbf{M}_j$-*shift invariant space* and the function φ_j is called *scaling function*. If the sequence $\{\mathbf{J}_l\}_{l\in\mathbb{N}}$ of regular matrices from Definition 2.1 is such that $|\det \mathbf{J}_l| = 2$, then the anisotropic PMRA is called *dyadic*.

The space V_j could be rewritten as the span of so-called orthogonal splines

$$f_{\boldsymbol{h}}^{\varphi_j}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \widehat{\varphi_j}(\boldsymbol{h} + \mathbf{M}_j^{\mathrm{T}}\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}(\boldsymbol{h}^{\mathrm{T}}+\boldsymbol{k}^{\mathrm{T}}\mathbf{M}_j)\boldsymbol{x}}, \quad \boldsymbol{h} \in G(\mathbf{M}_j^{\mathrm{T}}),$$

where $\widehat{\varphi_j}(\boldsymbol{k}) = \dfrac{1}{(2\pi)^2} \displaystyle\int_{\mathbb{T}^2} \varphi_j(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}$, $\boldsymbol{k} \in \mathbb{Z}^2$, are the Fourier coefficients of a function φ_j. Namely, the following statement is true.

Proposition 2.1. *(See* [5]*) For the $\mathbf{M}_j$-shift invariant space V_j it holds that*

$$V_j = \mathrm{span}\left\{ f_{\boldsymbol{h}}^{\varphi_j} \colon \boldsymbol{h} \in G(\mathbf{M}_j^{\mathrm{T}}) \right\}.$$

Definition 2.2. For the dyadic anisotropic PMRA the *wavelet space* W_j is the orthogonal complement of V_j in V_{j+1}, i.e.

$$W_j = V_{j+1} \ominus V_j, \quad j \in \mathbb{Z}_+.$$

From results of the paper [5] it follows that W_j is invariant with respect to the shifts $\mathrm{T}_{\boldsymbol{y}}$, $\boldsymbol{y} \in P(\mathbf{M}_j)$, and there exists a function ψ_j such that $W_j = \mathrm{span}\{\mathrm{T}_{\boldsymbol{y}}\psi_j \colon \boldsymbol{y} \in P(\mathbf{M}_j)\}$. Such a function ψ_j is called *wavelet*.

2.3. **Degree of trigonometric polynomials of two variables.** For a vector $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, we use l_q-*norm*

$$\|\boldsymbol{x}\|_{l_q} = \begin{cases} (|x_1|^q + |x_2|^q)^{1/q}, & 1 \le q < \infty, \\ \max\{|x_1|, |x_2|\}, & q = \infty, \end{cases}$$

and introduce the notion of q-*degree* for a trigonometric polynomial t of two variables as follows:

$$\deg_q t = \min\left\{ n \in \mathbb{Z}_+ \colon t(\boldsymbol{x}) = \sum_{\|\boldsymbol{k}\|_{l_q}\le n} c_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}}, \ c_{\boldsymbol{k}} \in \mathbb{C}, \ \boldsymbol{k} \in \mathbb{Z}^2 \right\}.$$

In this paper we restrict ourselves to considering only the cases $q = 1$ and $q = \infty$.

Let $\{\mathrm{e}^{\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\cdot} \colon \boldsymbol{k} \in \mathbb{Z}^2\}$ be the usual trigonometric monomial basis. We rewrite this basis by using single numeration $\{\mathrm{e}^{\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\cdot} \colon \boldsymbol{k} \in \mathbb{Z}^2\} = \{e_k \colon k \in \mathbb{N}\}$ so that the growth of $\deg_\infty e_k$ is
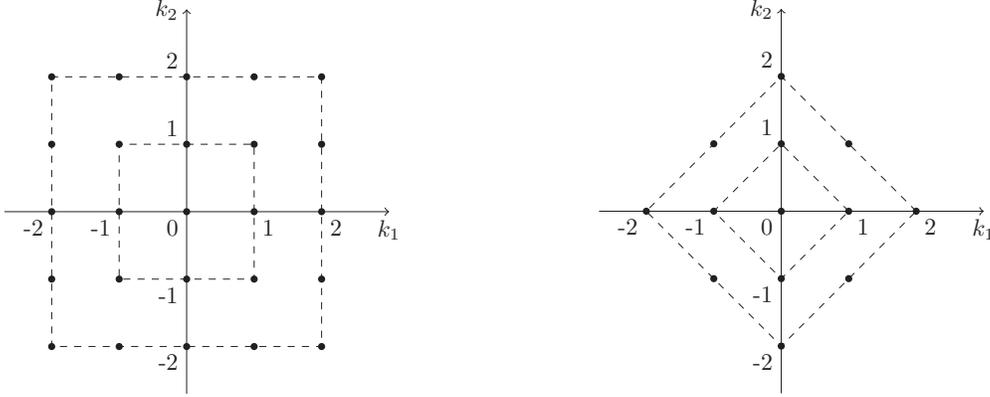
FIGURE 2. The method of selection of vectors $\boldsymbol{k} \in \mathbb{Z}^2$ so that the growth of $\deg_\infty \mathrm{e}^{\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\cdot}$ (left) and the growth of $\deg_1 \mathrm{e}^{\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\cdot}$ (right) are minimal.

minimal (see Figure 2). It is clear that $\deg_\infty e_k = n$ if $(2n-1)^2 + 1 \leq k \leq (2n+1)^2$, $n \in \mathbb{N}$, and consequently

$$\deg_\infty e_k = \left\lceil \frac{\sqrt{k}-1}{2} \right\rceil$$

where $\lceil a \rceil = \min\{n \in \mathbb{Z} \colon a \leq n\}$.

If we rewrite this basis so that the growth of $\deg_1 e_k$ is minimal (see Figure 2), then we obtain $\deg_1 e_k = n$ if $2n^2 - 2n + 2 \leq k \leq 2n^2 + 2n + 1$, $n \in \mathbb{N}$, and consequently

$$\deg_1 e_k = \left\lceil \frac{\sqrt{2k-1}-1}{2} \right\rceil .$$

Since the usual trigonometric basis is not a Schauder basis in $C(\mathbb{T}^2)$, analogously to the univariate case it is natural to try to find an orthogonal trigonometric Schauder basis $\{t_k \colon k \in \mathbb{N}\}$ in $C(\mathbb{T}^2)$ for which the growth of $\deg_q t_k$ is as small as possible.

2.4. **Particular polynomial functions.** Let us describe particular polynomial functions which will be useful to construct an orthonormal Schauder basis in the space $C(\mathbb{T}^2)$ with small growth of the polynomial degree.

Let $\mathbf{M}_j = \underbrace{\mathbf{J}_D \ldots \mathbf{J}_D}_{j} \mathbf{M}_0$, $j \in \mathbb{N}$, where $\mathbf{J}_D = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\mathbf{M}_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. After simple calculations we get $|\det \mathbf{M}_j| = 2^j$ and

(2.1)
$$\mathbf{M}_j = \begin{cases} (-4)^l \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), & j = 4l, \\ (-4)^l \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right), & j = 4l+1, \\ 2(-4)^l \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), & j = 4l+2, \\ 2(-4)^l \left(\begin{smallmatrix} -1 & -1 \\ 1 & -1 \end{smallmatrix}\right), & j = 4l+3, \end{cases} \quad l \in \mathbb{Z}_+.$$

The corresponding inverse matrix has the following form:

(2.2)
$$\mathbf{M}_j^{-1} = \begin{cases} \frac{(-1)^l}{\Delta_j} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), & j = 4l, \\ \frac{(-1)^l}{\Delta_j} \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right), & j = 4l+1, \\ \frac{(-1)^l}{\Delta_j} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), & j = 4l+2, \\ \frac{(-1)^l}{\Delta_j} \left(\begin{smallmatrix} -1 & 1 \\ -1 & -1 \end{smallmatrix}\right), & j = 4l+3, \end{cases} \quad l \in \mathbb{Z}_+,$$

where

$$\Delta_j = \begin{cases} \sqrt{|\det \mathbf{M}_j|} & \text{if } j \text{ is even,} \\ \sqrt{|\det \mathbf{M}_{j+1}|} & \text{if } j \text{ is odd,} \end{cases} \quad j \in \mathbb{Z}_+.$$

We also note that $\Delta_j \in \mathbb{N}$.

Further, for $0 < \alpha \le \frac{1}{14}$ we consider the piecewise polynomial

$$b_\alpha(x) = \begin{cases} 1 & \text{if } |x| \le \frac{1}{2} - \alpha, \\ \frac{3}{16\alpha^5} \left(\frac{1}{2} - |x|\right)^5 - \frac{5}{8\alpha^3} \left(\frac{1}{2} - |x|\right)^3 + \frac{15}{16\alpha} \left(\frac{1}{2} - |x|\right) + \frac{1}{2} & \text{if } \frac{1}{2} - \alpha < |x| < \frac{1}{2} + \alpha, \\ 0 & \text{if } |x| \ge \frac{1}{2} + \alpha, \end{cases}$$

and for $\boldsymbol{x} \in \mathbb{R}^2$ we set

$$B_\alpha(\boldsymbol{x}) = b_\alpha(x_1) b_\alpha(x_2),$$

$$\Phi_\alpha(\boldsymbol{x}) = \left(\sum_{\boldsymbol{z} \in \mathbb{Z}^2} B_\alpha(\boldsymbol{x} - \mathbf{J}_D^{\mathrm{T}} \boldsymbol{z}) \right) B_\alpha(\mathbf{J}_D^{-\mathrm{T}} \boldsymbol{x}),$$

$$\Psi_\alpha(\boldsymbol{x}) = \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{\omega}} \left(\sum_{\boldsymbol{z} \in \mathbb{Z}^2} B_\alpha(\boldsymbol{x} - \boldsymbol{\nu} - \mathbf{J}_D^{\mathrm{T}} \boldsymbol{z}) \right) \left(\sum_{\boldsymbol{z} \in \mathbb{Z}^2} B_\alpha(\mathbf{J}_D^{-\mathrm{T}} \boldsymbol{x} - \mathbf{J}_D^{\mathrm{T}} \boldsymbol{z}) \right) B_\alpha(\mathbf{J}_D^{-\mathrm{T}} \mathbf{J}_D^{-\mathrm{T}} \boldsymbol{x})$$

where $\boldsymbol{\omega} = \left(-\frac{1}{2}, -\frac{1}{2}\right)^{\mathrm{T}}$ and $\boldsymbol{\nu} = (-1, 0)^{\mathrm{T}}$.

By using Φ_α and Ψ_α, we now define functions

$$\varphi_{\alpha,j}(\boldsymbol{x}) = \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \Phi_\alpha\left(\mathbf{M}_j^{-\mathrm{T}} \boldsymbol{k}\right) \mathrm{e}^{\mathrm{i} \boldsymbol{k}^{\mathrm{T}} \boldsymbol{x}},$$

and

$$\psi_{\alpha,j}(\boldsymbol{x}) = \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \Psi_\alpha\left(\mathbf{M}_j^{-\mathrm{T}} \boldsymbol{k}\right) \mathrm{e}^{\mathrm{i} \boldsymbol{k}^{\mathrm{T}} \boldsymbol{x}}, \quad \boldsymbol{x} \in \mathbb{R}^2.$$

These functions constitute a particular case of a more general construction of the dyadic anisotropic PMRA which was considered in [1]. That is the functions $\varphi_{\alpha,j}$ are scaling functions and $\psi_{\alpha,j}$ are corresponding wavelets:

$$V_j = \mathrm{span}\left\{ \mathrm{T}_{\boldsymbol{y}} \varphi_{\alpha,j} \colon \boldsymbol{y} \in P(\mathbf{M}_j) \right\} \quad \text{and} \quad W_j = \mathrm{span}\left\{ \mathrm{T}_{\boldsymbol{y}} \psi_{\alpha,j} \colon \boldsymbol{y} \in P(\mathbf{M}_j) \right\}.$$

Further, we need to know how to orthonormalize basis elements of the spaces V_j and W_j without losing the shift invariance property of these spaces. To this end, we set (see Figure 3)

$$\Phi_\alpha^\perp(\boldsymbol{x}) = \frac{\Phi_\alpha(\boldsymbol{x})}{\left(\displaystyle\sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \le 2}} |\Phi_\alpha(\boldsymbol{x} - \boldsymbol{z})|^2 \right)^{1/2}}, \quad \Psi_\alpha^\perp(\boldsymbol{x}) = \frac{\Psi_\alpha(\boldsymbol{x})}{\left(\displaystyle\sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \le 2}} |\Psi_\alpha(\boldsymbol{x} - \boldsymbol{z})|^2 \right)^{1/2}}$$

and define the following two functions:

$$\varphi_{\alpha,j}^\perp(\boldsymbol{x}) = \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \Phi_\alpha^\perp\left(\mathbf{M}_j^{-\mathrm{T}} \boldsymbol{k}\right) \mathrm{e}^{\mathrm{i} \boldsymbol{k}^{\mathrm{T}} \boldsymbol{x}}$$

and

$$\psi_{\alpha,j}^\perp(\boldsymbol{x}) = \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \Psi_\alpha^\perp\left(\mathbf{M}_j^{-\mathrm{T}} \boldsymbol{k}\right) \mathrm{e}^{\mathrm{i} \boldsymbol{k}^{\mathrm{T}} \boldsymbol{x}}, \quad \boldsymbol{x} \in \mathbb{R}^2.$$
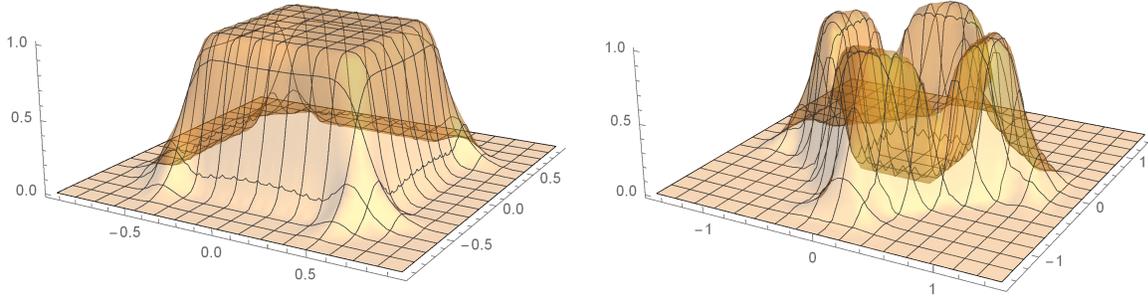
FIGURE 3. Plot of functions $\Phi_\alpha^\perp$ (left) and $|\Psi_\alpha^\perp|$ (right) for $\alpha = \frac{1}{14}$.

In view of the results [5, Corollaries 3.6 and 3.7], it follows that the shifts $\mathrm{T}_{\boldsymbol{y}} \varphi_{\alpha,j}^\perp$ and $\mathrm{T}_{\boldsymbol{y}} \psi_{\alpha,j}^\perp$, $\boldsymbol{y} \in P(\mathbf{M}_j)$, constitute orthonormal basis elements for the spaces V_j and W_j respectively:

$$V_j = \operatorname{span}\left\{\mathrm{T}_{\boldsymbol{y}} \varphi_{\alpha,j}^\perp \colon \boldsymbol{y} \in P(\mathbf{M}_j)\right\} \quad \text{and} \quad W_j = \operatorname{span}\left\{\mathrm{T}_{\boldsymbol{y}} \psi_{\alpha,j}^\perp \colon \boldsymbol{y} \in P(\mathbf{M}_j)\right\}.$$

2.5. **The main result.** For any set $A \subset \mathbb{Z}^2$ we consider a set of trigonometric polynomials

$$\mathcal{T}_A = \left\{t \colon t(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in A} c_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}}, \ c_{\boldsymbol{k}} \in \mathbb{C}\right\},$$

and for any function $f \in C(\mathbb{T}^2)$ we denote

$$E_A(f)_\infty = \inf_{t \in \mathcal{T}_A} \|f - t\|_\infty.$$

The quantity $E_A(f)_\infty$ is called the best approximation of the function f by trigonometric polynomials from the set $\mathcal{T}_A$ in the uniform metric.

Further, for arbitrary $\varepsilon > 0$ we set

$$\lambda = \begin{cases} \frac{1}{14} & \text{if } \varepsilon \geq \frac{6}{7}, \\ \frac{\varepsilon}{12} & \text{if } 0 < \varepsilon < \frac{6}{7}. \end{cases}$$

Definition 2.3. For given $\varepsilon > 0$ the set of polynomials $\{t_k \colon k \in \mathbb{N}\}$ is defined by

$$t_1 = 1$$

and for $k = 2^j + 1, \ldots, 2^{j+1}$, $j \in \mathbb{Z}_+$, by

$$t_k = \mathrm{T}_{\boldsymbol{y}_k} \psi_{\lambda,j}^\perp, \quad \boldsymbol{y}_k \in P(\mathbf{M}_j).$$

For the polynomial set $\{t_k \colon k \in \mathbb{N}\}$ from Definition 2.3 and any function $f \in C(\mathbb{T}^2)$ we define the operator

$$(2.3) \qquad S_\mu f = \sum_{k=1}^{\mu} \langle f, t_k \rangle \, t_k$$

where $\langle f, t_k \rangle = \dfrac{1}{(2\pi)^2} \displaystyle\int_{\mathbb{T}^2} f(\boldsymbol{x}) \overline{t_k(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x}$.

Theorem 2.1. *For given $\varepsilon > 0$ the polynomial system $\{t_k \colon k \in \mathbb{N}\}$ from Definition 2.3 is an orthonormal Schauder basis in $C(\mathbb{T}^2)$ such that*

$$(2.4) \qquad \deg_\infty t_k \leq (2 + \varepsilon)\left\lceil \frac{\sqrt{k} - 1}{2} \right\rceil$$
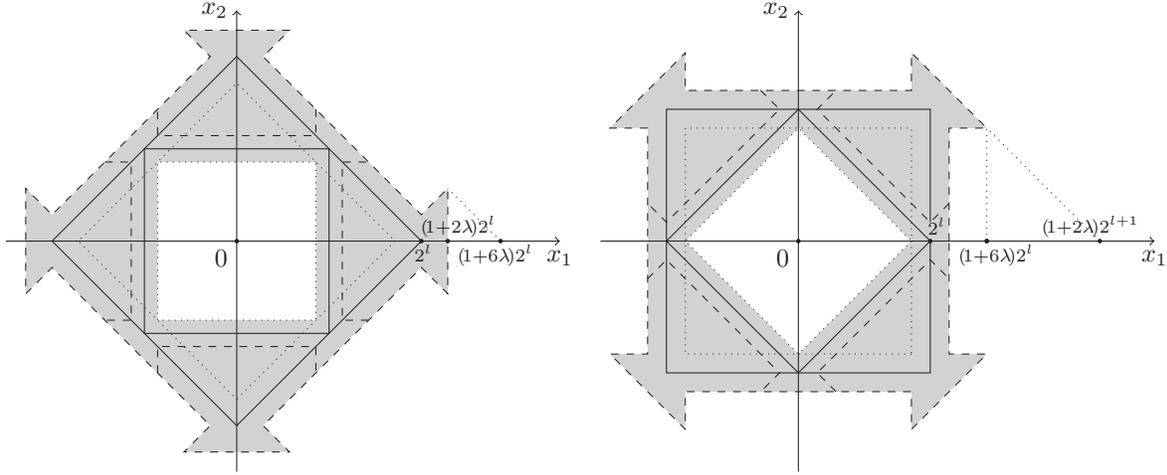
FIGURE 4. The support of the functions $\Psi_\lambda^\perp(\mathbf{M}_{2l}^{-\mathrm{T}}\cdot)$ (left) and $\Psi_\lambda^\perp(\mathbf{M}_{2l+1}^{-\mathrm{T}}\cdot)$ (right), $l \in \mathbb{Z}_+$.

and

$$\deg_1 t_k \leq (2+\varepsilon)\left\lceil \frac{\sqrt{2k-1}-1}{2}\right\rceil. \tag{2.5}$$

Moreover, for all $f \in C(\mathbb{T}^2)$ the following inequality is true:

$$\|f - S_\mu f\|_\infty \leq (1 + C(\varepsilon))\, E_{A_\mu}(f)_\infty \tag{2.6}$$

where $A_\mu = \left\{ \boldsymbol{k} \in \mathbb{Z}^2 \colon \mathbf{M}_j^{-\mathrm{T}}\boldsymbol{k} \in \left[-\frac{1}{2}+\lambda, \frac{1}{2}-\lambda\right]^2 \right\}$ for $2^j \leq \mu < 2^{j+1}$, $j \in \mathbb{Z}_+$, and $C(\varepsilon) > 0$ is a positive constant which depends only on ε.

Proof. The orthonormality of the polynomial system $\{t_k \colon k \in \mathbb{N}\}$ from Definition 2.3 follows from the construction of this system which was discussed in the previous subsection.

We prove the degree inequalities (2.4) and (2.5). Note that these inequalities are trivial for $k = 1$. For $k \geq 2$ the polynomial system $\{t_k \colon k \in \mathbb{N}\}$ is generated by basis elements of wavelet spaces W_j, $j \in \mathbb{Z}_+$. For this reason, depending on the values of the parameter j, we consider two cases.

Case I. Let $j = 2l$, $l \in \mathbb{Z}_+$. In this case for $2^{2l} < k \leq 2^{2l+1}$ we have (see Figure 4)

$$\deg_\infty t_k = \deg_\infty \psi_{\lambda,2l}^\perp \leq (1+2\lambda)2^l$$

and

$$\deg_1 t_k = \deg_1 \psi_{\lambda,2l}^\perp \leq (1+6\lambda)2^l.$$

Hence, to prove (2.4) and (2.5), it is sufficient to show that for $2^{2l} < k \leq 2^{2l+1}$ it holds

$$(1+2\lambda)2^l \leq (2+\varepsilon)\left\lceil \frac{\sqrt{k}-1}{2}\right\rceil \tag{2.7}$$

and

$$(1+6\lambda)2^l \leq (2+\varepsilon)\left\lceil \frac{\sqrt{2k-1}-1}{2}\right\rceil. \tag{2.8}$$

For $l = 0$ or $l = 1$ the inequalities (2.7) and (2.8) are easily verified. If $l \geq 2$, then for $2^{2l} < k \leq 2^{2l+1}$ we have

$$(2 + \varepsilon) \left\lceil \frac{\sqrt{k} - 1}{2} \right\rceil \geq (2 + \varepsilon) \left\lceil \frac{\sqrt{2^{2l}} - 1}{2} \right\rceil = (2 + \varepsilon) \, 2^{l-1} \geq (2 + 12\lambda) 2^{l-1} > (1 + 2\lambda) 2^l$$

and

$$(2 + \varepsilon) \left\lceil \frac{\sqrt{2k - 1} - 1}{2} \right\rceil \geq (2 + \varepsilon) \left\lceil \frac{\sqrt{2(2^{2l} + 1) - 1} - 1}{2} \right\rceil$$

$$\geq (2 + \varepsilon) \left\lceil \frac{\sqrt{2^{2l+1}} - 1}{2} \right\rceil \geq (2 + \varepsilon) \left(\sqrt{2} - \frac{1}{2^l} \right) 2^{l-1}$$

$$> (2 + \varepsilon) \, 2^{l-1} \geq (2 + 12\lambda) 2^{l-1} = (1 + 6\lambda) 2^l.$$

Case II. Let $j = 2l + 1$, $l \in \mathbb{Z}_+$. In this case for $2^{2l+1} < k \leq 2^{2(l+1)}$ we have (see Figure 4)

$$\deg_\infty t_k = \deg_\infty \psi_{\lambda, 2l+1}^\perp \leq (1 + 6\lambda) 2^l$$

and

$$\deg_1 t_k = \deg_1 \psi_{\lambda, 2l+1}^\perp \leq (1 + 2\lambda) 2^{l+1}.$$

Hence, to prove (2.4) and (2.5), it is sufficient to show that for $2^{2l+1} < k \leq 2^{2(l+1)}$ it holds

(2.9)
$$(1 + 6\lambda) 2^l \leq (2 + \varepsilon) \left\lceil \frac{\sqrt{k} - 1}{2} \right\rceil$$

and

(2.10)
$$(1 + 2\lambda) 2^{l+1} \leq (2 + \varepsilon) \left\lceil \frac{\sqrt{2k - 1} - 1}{2} \right\rceil.$$

Similarly to the previous case, if $l = 0$ or $l = 1$, then inequalities (2.9) and (2.10) are easily verified. If $l \geq 2$, then for $2^{2l+1} < k \leq 2^{2(l+1)}$ we have

$$(2 + \varepsilon) \left\lceil \frac{\sqrt{k} - 1}{2} \right\rceil \geq (2 + \varepsilon) \left\lceil \frac{\sqrt{2^{2l+1}} - 1}{2} \right\rceil$$

$$\geq (2 + \varepsilon) \left(\sqrt{2} - \frac{1}{2^l} \right) 2^{l-1} > (2 + \varepsilon) \, 2^{l-1}$$

$$\geq (2 + 12\lambda) 2^{l-1} = (1 + 6\lambda) 2^l$$

and

$$(2 + \varepsilon) \left\lceil \frac{\sqrt{2k - 1} - 1}{2} \right\rceil \geq (2 + \varepsilon) \left\lceil \frac{\sqrt{2(2^{2l+1} + 1) - 1} - 1}{2} \right\rceil$$

$$\geq (2 + \varepsilon) \left\lceil \frac{\sqrt{2^{2(l+1)}} - 1}{2} \right\rceil = (2 + \varepsilon) \, 2^l$$

$$\geq (2 + 12\lambda) 2^l > (1 + 2\lambda) 2^{l+1}.$$

Hence, inequalities (2.4) and (2.5) are proved.

Finally, we prove the approximation property (2.6) from which it follows that the polynomial system $\{t_k \colon k \in \mathbb{N}\}$ from Definition 2.3 is a Schauder basis in $C(\mathbb{T}^2)$. In view of Proposition 2.1 we can see that $\mathcal{T}_{A_\mu} \subseteq \operatorname{span} \{t_k \colon k = 1, \ldots, \mu\}$ which implies the equality $S_\mu t = t$ for arbitrary $t \in \mathcal{T}_{A_\mu}$. Thus, choosing $t^* \in \mathcal{T}_{A_\mu}$ as the polynomial of the best

approximation of the function $f \in C(\mathbb{T}^2)$ and applying Theorem 3.1 (see next section), we get

$$\|f - S_\mu f\|_\infty = \|f - t^* + S_\mu(t^* - f)\|_\infty \leq \|f - t^*\|_\infty + \|S_\mu(t^* - f)\|_\infty$$
$$\leq \left(1 + \|S_\mu\|_{C(\mathbb{T}^2) \to C(\mathbb{T}^2)}\right) \|f - t^*\|_\infty \leq (1 + C(\varepsilon))\, E_{A_\mu}(f)_\infty.$$

$\square$

3. Norm of the orthogonal projection operator

First, we formulate and prove some auxiliary results.

Lemma 3.1. *For any function $f \in C^{\boldsymbol{r}}(\mathbb{T}^2)$, $\boldsymbol{r} \in \mathbb{Z}_+^2$, and any vector $\boldsymbol{k} \in \mathbb{Z}^2$ the following equality is true:*

$$(3.1) \qquad \widehat{f}(\boldsymbol{k}) = \begin{cases} \dfrac{1}{(2\pi)^2} \displaystyle\int_{\mathbb{T}^2} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, & k_1 = k_2 = 0, \\[2.5ex] \dfrac{1}{(2\pi)^2\, (\mathrm{i}k_1)^{r_1}} \displaystyle\int_{\mathbb{T}^2} \dfrac{\partial^{r_1} f}{\partial x_1^{r_1}}(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, & k_1 \neq 0,\ k_2 = 0, \\[2.5ex] \dfrac{1}{(2\pi)^2\, (\mathrm{i}k_2)^{r_2}} \displaystyle\int_{\mathbb{T}^2} \dfrac{\partial^{r_2} f}{\partial x_2^{r_2}}(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, & k_1 = 0,\ k_2 \neq 0, \\[2.5ex] \dfrac{1}{(2\pi)^2\, \mathrm{i}^{\|\boldsymbol{r}\|_{l_1}} k_1^{r_1} k_2^{r_2}} \displaystyle\int_{\mathbb{T}^2} \dfrac{\partial^{\|\boldsymbol{r}\|_{l_1}} f}{\partial x_1^{r_1} \partial x_2^{r_2}}(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, & k_1 k_2 \neq 0. \end{cases}$$

Note that in the one-dimensional case the corresponding result is well-known (see, e.g. [4, Chap. 19]). For the functions of two variables the proof is similar to the one-dimensional case.

Remark 3.1. For any vector $\boldsymbol{r} \in \mathbb{N}^2$ the equality (3.1) can be extended to all $\boldsymbol{k} \in \mathbb{R}^2$. In order to do so it is sufficient to impose the additional condition

$$(3.2) \qquad \frac{\partial^{\|\boldsymbol{l}\|_1} f}{\partial x_1^{l_1} \partial x_2^{l_2}}(-\pi, x_2) = \frac{\partial^{\|\boldsymbol{l}\|_1} f}{\partial x_1^{l_1} \partial x_2^{l_2}}(\pi, x_2) = \frac{\partial^{\|\boldsymbol{l}\|_1} f}{\partial x_1^{l_1} \partial x_2^{l_2}}(x_1, -\pi) = \frac{\partial^{\|\boldsymbol{l}\|_1} f}{\partial x_1^{l_1} \partial x_2^{l_2}}(x_1, \pi) = 0$$

where $\boldsymbol{l} \in \mathbb{Z}_+^2$ and $0 \leq l_j \leq r_j - 1$, $j \in \{1, 2\}$.

Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary function that satisfies the condition

$$(3.3) \qquad \sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \leq 2}} f^2(\boldsymbol{x} + \boldsymbol{z}) > 0, \quad \boldsymbol{x} \in \operatorname{supp} f,$$

and

$$f^\perp(\boldsymbol{x}) = \begin{cases} \dfrac{f(\boldsymbol{x})}{\left(\displaystyle\sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \leq 2}} f^2(\boldsymbol{x} + \boldsymbol{z}) \right)^{1/2}}, & \boldsymbol{x} \in \operatorname{supp} f, \\[4ex] 0, & \text{else.} \end{cases}$$

Lemma 3.2. *If $f \in C^{\boldsymbol{r}}(\mathbb{R}^2)$, $\boldsymbol{r} \in \mathbb{Z}_+^2$, then $f^\perp \in C^{\boldsymbol{r}}(\mathbb{R}^2)$ too.*

Proof. For $\boldsymbol{x} \in \operatorname{supp} f$ we have

$$\frac{\partial f^{\perp}}{\partial x_1}(\boldsymbol{x}) = \frac{\dfrac{\partial f}{\partial x_1}(\boldsymbol{x}) \displaystyle\sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \leq 2}} f^2(\boldsymbol{x}+\boldsymbol{z}) - f(\boldsymbol{x}) \displaystyle\sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \leq 2}} f(\boldsymbol{x}+\boldsymbol{z})\dfrac{\partial f}{\partial x_1}(\boldsymbol{x}+\boldsymbol{z})}{\left(\displaystyle\sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \leq 2}} f^2(\boldsymbol{x}+\boldsymbol{z}) \right)^{3/2}}$$

$$= \frac{f_1(\boldsymbol{x})}{\left(\displaystyle\sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \leq 2}} f^2(\boldsymbol{x}+\boldsymbol{z}) \right)^{3/2}}.$$

It is clear that $f_1 \in C^{(r_1-1, r_2)}(\mathbb{R}^2)$ and $\operatorname{supp} f_1 \subseteq \operatorname{supp} f$.

Continuing to differentiate, for $\boldsymbol{x} \in \operatorname{supp} f$ we get

$$\frac{\partial^{\|\boldsymbol{r}\|_{l_1}} f^{\perp}}{\partial x_1^{r_1} \partial x_2^{r_2}}(\boldsymbol{x}) = \frac{f_{\|\boldsymbol{r}\|_{l_1}}(\boldsymbol{x})}{\left(\displaystyle\sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \leq 2}} f^2(\boldsymbol{x}+\boldsymbol{z}) \right)^{\|\boldsymbol{r}\|_{l_1}+1/2}}$$

where $f_{\|\boldsymbol{r}\|_{l_1}} \in C(\mathbb{R}^2)$ and $\operatorname{supp} f_{\|\boldsymbol{r}\|_{l_1}} \subseteq \operatorname{supp} f$. Hence,

$$\frac{\partial^{\|\boldsymbol{r}\|_{l_1}} f^{\perp}}{\partial x_1^{r_1} \partial x_2^{r_2}}(\boldsymbol{x}) = \begin{cases} \dfrac{f_{\|\boldsymbol{r}\|_{l_1}}(\boldsymbol{x})}{\left(\displaystyle\sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \leq 2}} f^2(\boldsymbol{x}+\boldsymbol{z}) \right)^{\|\boldsymbol{r}\|_{l_1}+1/2}}, & \boldsymbol{x} \in \operatorname{supp} f_{\|\boldsymbol{r}\|_{l_1}}, \\ 0, & \text{else.} \end{cases}$$

Thus, since $\displaystyle\sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^2 \\ \|\boldsymbol{z}\|_{l_1} \leq 2}} f^2(\boldsymbol{x}+\boldsymbol{z}) > 0$ for $\boldsymbol{x} \in \operatorname{supp} f_{\|\boldsymbol{r}\|_{l_1}}$, we get $\dfrac{\partial^{\|\boldsymbol{r}\|_{l_1}} f^{\perp}}{\partial x_1^{r_1} \partial x_2^{r_2}} \in C(\mathbb{R}^2)$. $\square$

Corollary 3.1. *For $0 < \alpha \leq \frac{1}{14}$ the functions $\Phi_\alpha^{\perp}$ and $|\Psi_\alpha^{\perp}|$ belong to the space $C^{(2,2)}(\mathbb{R}^2)$.*

Proof. It is easy to verify that function b_α has a continuous second derivative and consequently $B_\alpha \in C^{(2,2)}(\mathbb{R}^2)$. As it is shown in the paper [1, Theorem 4.3 and 4.6], in this case the functions Φ_α and $|\Psi_\alpha|$ also belong to the space $C^{(2,2)}(\mathbb{R}^2)$. Furthermore, these functions satisfy condition (3.3). Thus, in view of Lemma 3.2, we get the assertion. $\square$

Now, we formulate and prove the statement which generalizes a one-dimensional result from [12, Chap. 4].

Lemma 3.3. *For any polynomial t, 2π-periodic in each variable and any matrix $\mathbf{M}_j$, $j \in \mathbb{Z}_+$, which satisfies (2.1), it holds that*

(3.4) $$\|t\|_1 \leq \frac{1}{|\det \mathbf{M}_j|} \sup_{\boldsymbol{x} \in \mathbb{T}^2} \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |t(\boldsymbol{x} - 2\pi\boldsymbol{y})|.$$
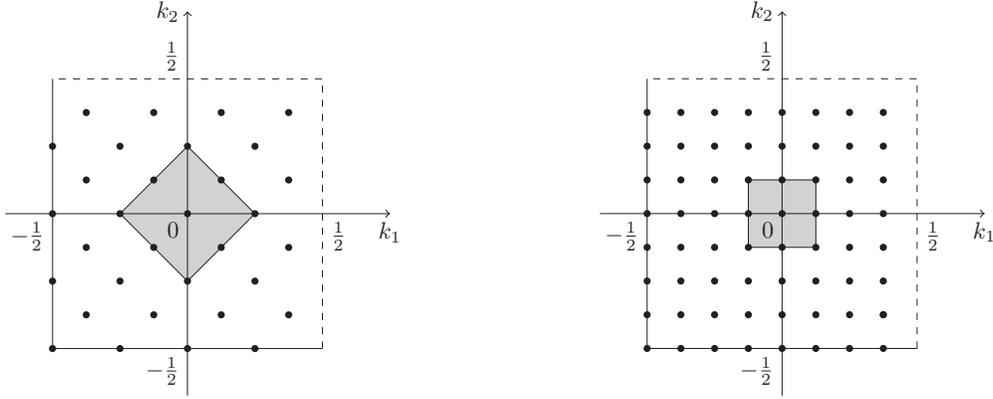
FIGURE 5. The set $\square_\mathbf{0}$ for the matrix $\mathbf{M}_5$ (left) and $\mathbf{M}_6$ (right).

Proof. Let $\square_{\boldsymbol{y}} = \left\{ \boldsymbol{x} - \boldsymbol{y} \colon \boldsymbol{x} = \mathbf{M}_j^{-1}\boldsymbol{\xi}, \ \|\boldsymbol{\xi}\|_{l_\infty} \leq 1 \right\}$, $\boldsymbol{y} \in P(\mathbf{M}_j)$. It is clear that $\square_{\boldsymbol{y}}$ is a shift of the set $\square_\mathbf{0}$ (see Figure 5) by the vector $\boldsymbol{y} \in P(\mathbf{M}_j)$. Moreover,

$$\mathrm{vol}\left(\bigcup_{\boldsymbol{y} \in P(\mathbf{M}_j)} \square_{\boldsymbol{y}} \right) = 4\,\mathrm{vol}\left(\left[-\frac{1}{2}, \frac{1}{2} \right]^2 \right) = 4.$$

Thus, in view of $|P(\mathbf{M}_j)| = |\det \mathbf{M}_j|$, we get $\mathrm{vol}(\square_\mathbf{0}) = \dfrac{4}{|\det \mathbf{M}_j|}$.

Since the polynomial t is 2π-periodic in each variable, we can write

$$\|t\|_1 = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} |t(\boldsymbol{x})| \mathrm{d}\boldsymbol{x}$$

$$= \frac{1}{4} \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} \int_{\square_{\boldsymbol{y}}} |t(2\pi\boldsymbol{x})| \mathrm{d}\boldsymbol{x} = \frac{1}{4} \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} \int_{\square_\mathbf{0}} |t(2\pi(\boldsymbol{x}-\boldsymbol{y}))| \mathrm{d}\boldsymbol{x}$$

$$\leq \frac{1}{4}\,\mathrm{vol}(\square_\mathbf{0}) \sup_{\boldsymbol{x} \in \square_\mathbf{0}} \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |t(2\pi(\boldsymbol{x}-\boldsymbol{y}))| = \frac{1}{|\det \mathbf{M}_j|} \sup_{\boldsymbol{x} \in \mathbb{T}^2} \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |t(\boldsymbol{x} - 2\pi\boldsymbol{y})|.$$

$\square$

Let us introduce the set (see Figure 6)

$$\Omega_j = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \colon \mathbf{M}_j^{-\mathrm{T}}\boldsymbol{x} \in [-2,2)^2 \right\}.$$

Note that this set contains $\mathrm{supp}\,\Phi_\alpha^\perp(\mathbf{M}_j^{-\mathrm{T}}\cdot)$ and $\mathrm{supp}\,\Psi_\alpha^\perp(\mathbf{M}_j^{-\mathrm{T}}\cdot)$. Further, we need a generalization of the well-known equality:

$$(3.5) \qquad \sum_{k=-M}^{M-1} \mathrm{e}^{\pi \mathrm{i} k \frac{m}{M}} = \begin{cases} 2M, & m = 2Ms, \ s \in \mathbb{Z}, \\ 0, & \text{else}, \end{cases}$$

where $m \in \mathbb{Z}$ and $M \in \mathbb{N}$.

Lemma 3.4. *For any vector $\boldsymbol{m} \in \mathbb{Z}^2$ and any matrix $\mathbf{M}_j$, $j \in \mathbb{Z}_+$, which satisfies (2.1), it holds that*

$$(3.6) \qquad \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} \mathrm{e}^{\frac{\pi}{2}\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\left(\mathbf{M}_j^{-1}\boldsymbol{m}\right)} = \begin{cases} 16|\det \mathbf{M}_j|, & \boldsymbol{m} = 4\mathbf{M}_j\boldsymbol{s}, \ \boldsymbol{s} \in \mathbb{Z}^2, \\ 0, & \text{else}. \end{cases}$$
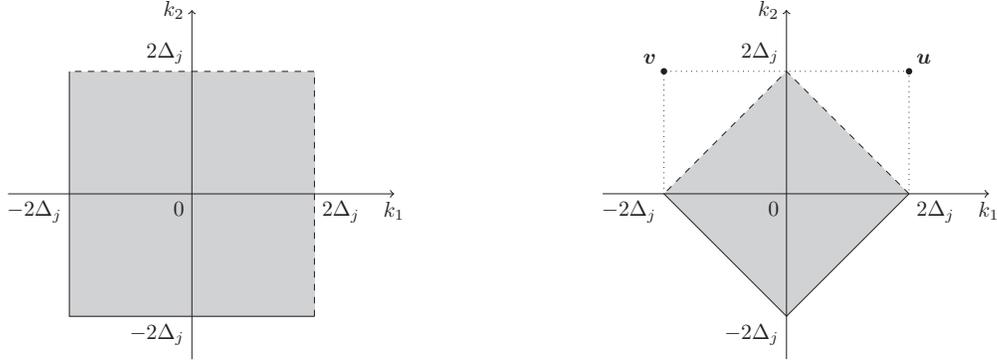
FIGURE 6. The set Ω_j for $j = 8l$ (left) and $j = 8l + 1$ (right), $l \in \mathbb{Z}_+$.

Proof. If $\boldsymbol{m} = 4\mathbf{M}_j\boldsymbol{s}$, $\boldsymbol{s} \in \mathbb{Z}^2$, then we get

$$\sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} \mathrm{e}^{\frac{\pi}{2}\mathrm{i}\boldsymbol{k}^{\mathrm{T}}(\mathbf{M}_j^{-1}\boldsymbol{m})} = \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}^{\mathrm{T}}(\mathbf{M}_j^{-1}\mathbf{M}_j\boldsymbol{s})}$$

$$= \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{s}} = \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} 1 = 16|\det \mathbf{M}_j|.$$

Let $\boldsymbol{m} \neq 4\mathbf{M}_j\boldsymbol{s}$, $\boldsymbol{s} \in \mathbb{Z}^2$. Then $\boldsymbol{p} = \mathbf{M}_j^{-1}\boldsymbol{m} \neq 4\boldsymbol{s}$, $\boldsymbol{s} \in \mathbb{Z}^2$. Depending on the values of the parameter j, we consider several cases.

Case I. Let $j = 8l$, $l \in \mathbb{Z}_+$. In this case, in view of (2.2), we have

$$\boldsymbol{p} = (p_1, p_2)^{\mathrm{T}} = \left(\frac{m_1}{\Delta_j}, \frac{m_2}{\Delta_j}\right)^{\mathrm{T}}.$$

Since $\boldsymbol{p} \neq 4\boldsymbol{s}$, $\boldsymbol{s} \in \mathbb{Z}^2$, without loss of generality we can assume that $p_1 \not\equiv 0 \pmod 4$, i.e. $m_1 \neq 4n\Delta_j$, $n \in \mathbb{Z}$.

Taking into consideration the view of the set Ω_j (see Figure 6) and using equality (3.5), we obtain

$$\sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} \mathrm{e}^{\frac{\pi}{2}\mathrm{i}\boldsymbol{k}^{\mathrm{T}}(\mathbf{M}_j^{-1}\boldsymbol{m})} = \sum_{k_1=-2\Delta_j}^{2\Delta_j-1} \mathrm{e}^{\frac{\pi}{2}\mathrm{i}k_1 p_1} \sum_{k_2=-2\Delta_j}^{2\Delta_j-1} \mathrm{e}^{\frac{\pi}{2}\mathrm{i}k_2 p_2} = \sum_{k_1=-2\Delta_j}^{2\Delta_j-1} \mathrm{e}^{\pi\mathrm{i}k_1\frac{m_1}{2\Delta_j}} \sum_{k_2=-2\Delta_j}^{2\Delta_j-1} \mathrm{e}^{\frac{\pi}{2}\mathrm{i}k_2 p_2} = 0$$

because $m_1 \neq 4n\Delta_j$, $n \in \mathbb{Z}$.

Case II. Let $j = 8l + 1$, $l \in \mathbb{Z}_+$. In this case, in view of (2.2), we have

$$\boldsymbol{p} = (p_1, p_2)^{\mathrm{T}} = \left(\frac{m_1 + m_2}{\Delta_j}, \frac{m_2 - m_1}{\Delta_j}\right)^{\mathrm{T}}.$$

Since $\boldsymbol{p} \neq 4\boldsymbol{s}$, $\boldsymbol{s} \in \mathbb{Z}^2$, without loss of generality we can assume that $p_1 \not\equiv 0 \pmod 4$, i.e. $m_1 + m_2 \neq 4n\Delta_j$, $n \in \mathbb{Z}$.

It is easy to verify that for the vector $\boldsymbol{u} = (2\Delta_j, 2\Delta_j)^{\mathrm{T}}$ it holds that

$$(3.7) \qquad\qquad \mathrm{e}^{\frac{\pi}{2}\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{p}} = \mathrm{e}^{\frac{\pi}{2}\mathrm{i}(\boldsymbol{k}+\boldsymbol{u})^{\mathrm{T}}\boldsymbol{p}}, \quad \boldsymbol{k} \in \mathbb{Z}^2.$$

Analogously, for the vector $\boldsymbol{v} = (-2\Delta_j, 2\Delta_j)^{\mathrm{T}}$ we have

$$(3.8) \qquad\qquad \mathrm{e}^{\frac{\pi}{2}\mathrm{i}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{p}} = \mathrm{e}^{\frac{\pi}{2}\mathrm{i}(\boldsymbol{k}+\boldsymbol{v})^{\mathrm{T}}\boldsymbol{p}}, \quad \boldsymbol{k} \in \mathbb{Z}^2.$$

Thus, using (3.7) and (3.8), we can write (see Figure 6)

$$(3.9) \qquad\qquad \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} \mathrm{e}^{\frac{\pi}{2}\mathrm{i}\boldsymbol{k}^{\mathrm{T}}(\mathbf{M}_j^{-1}\boldsymbol{m})} = \sum_{\boldsymbol{k} \in \Pi_j \cap \mathbb{Z}^2} \mathrm{e}^{\frac{\pi}{2}\mathrm{i}\boldsymbol{k}^{\mathrm{T}}(\mathbf{M}_j^{-1}\boldsymbol{m})}$$

where $\Pi_j = [-2\Delta_j, 2\Delta_j) \times [0, 2\Delta_j)$.

Thus, in view of (3.5) and (3.9), we get

$$\sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} e^{\frac{\pi}{2} i \boldsymbol{k}^T (\mathbf{M}_j^{-1} \boldsymbol{m})} = \sum_{k_1 = -2\Delta_j}^{2\Delta_j - 1} e^{\frac{\pi}{2} i k_1 p_1} \sum_{k_2 = 0}^{2\Delta_j - 1} e^{\frac{\pi}{2} i k_2 p_2} = \sum_{k_1 = -2\Delta_j}^{2\Delta_j - 1} e^{\pi i k_1 \frac{m_1 + m_2}{2\Delta_j}} \sum_{k_2 = 0}^{2\Delta_j - 1} e^{\frac{\pi}{2} i k_2 p_2} = 0$$

because $m_1 + m_2 \neq 4n\Delta_j$, $n \in \mathbb{Z}$.

Case III. For $j = 8l + s$, $2 \leq s \leq 7$, we have a similar proof to that in case I (if s is even) and case II (if s is odd). $\qquad \square$

Lemma 3.5. *For the functions $\varphi_{\alpha,j}^\perp$ the following inequality is true:*

$$(3.10) \qquad \left\| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |\varphi_{\alpha,j}^\perp(\cdot - 2\pi \boldsymbol{y})| \right\|_\infty \leq C_1(\alpha) \sqrt{|\det \mathbf{M}_j|}, \quad C_1(\alpha) > 0.$$

Proof. By the definition of the function $\varphi_{\alpha,j}^\perp$ we have

$$I = \left\| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |\varphi_{\alpha,j}^\perp(\cdot - 2\pi \boldsymbol{y})| \right\|_\infty$$

$$= \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sup_{\boldsymbol{x} \in \mathbb{T}^2} \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} \left| \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} \Phi_\alpha^\perp \left(\mathbf{M}_j^{-T} \boldsymbol{k} \right) e^{i \boldsymbol{k}^T (\boldsymbol{x} - 2\pi \boldsymbol{y})} \right|$$

$$= \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sup_{\boldsymbol{x} \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2} \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} \left| \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} g_\alpha \left(\frac{\pi}{2} \mathbf{M}_j^{-T} \boldsymbol{k} \right) e^{2\pi i \boldsymbol{k}^T (\boldsymbol{x} - \boldsymbol{y})} \right|$$

where $g_\alpha(\boldsymbol{x}) = \Phi_\alpha^\perp \left(\frac{2\boldsymbol{x}}{\pi} \right)$, $\boldsymbol{x} \in \mathbb{T}^2$.

The function in the supremum norm is periodic with respect to any vector $\boldsymbol{y} \in P(\mathbf{M}_j)$. Thus, for this function it suffices to consider the supremum over the set $\square_{\mathbf{0}}$. Hence, we can write

$$I = \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sup_{\boldsymbol{x} \in \square_{\mathbf{0}}} \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} \left| \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} g_\alpha \left(\frac{\pi}{2} \mathbf{M}_j^{-T} \boldsymbol{k} \right) e^{2\pi i \boldsymbol{k}^T (\boldsymbol{x} - \boldsymbol{y})} \right|$$

$$= \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sup_{\|\boldsymbol{\xi}\|_{l_\infty} \leq 1} \sum_{\boldsymbol{h} \in G(\mathbf{M}_j)} \left| \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} g_\alpha \left(\frac{\pi}{2} \mathbf{M}_j^{-T} \boldsymbol{k} \right) e^{2\pi i \boldsymbol{k}^T (\mathbf{M}_j^{-1} \boldsymbol{\xi})} e^{-2\pi i \boldsymbol{k}^T (\mathbf{M}_j^{-1} \boldsymbol{h})} \right|$$

$$= \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sup_{\|\boldsymbol{\xi}\|_{l_\infty} \leq 1} \sum_{\boldsymbol{h} \in G(\mathbf{M}_j)} \left| \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} g_\alpha \left(\frac{\pi}{2} \mathbf{M}_j^{-T} \boldsymbol{k} \right) e^{i(4\boldsymbol{\xi})^T \left(\frac{\pi}{2} \mathbf{M}_j^{-T} \boldsymbol{k} \right)} e^{-i(4\boldsymbol{h})^T \left(\frac{\pi}{2} \mathbf{M}_j^{-T} \boldsymbol{k} \right)} \right|.$$

Let $f^{\boldsymbol{\xi}}(\boldsymbol{x}) = g_\alpha(\boldsymbol{x}) e^{i(4\boldsymbol{\xi})^T \boldsymbol{x}}$, $\boldsymbol{x} \in \mathbb{T}^2$. Then, expanding $f^{\boldsymbol{\xi}}$ in Fourier series

$$f^{\boldsymbol{\xi}}(\boldsymbol{x}) = \sum_{\boldsymbol{z} \in \mathbb{Z}^2} \widehat{f^{\boldsymbol{\xi}}}(\boldsymbol{z}) e^{i \boldsymbol{z}^T \boldsymbol{x}},$$

and applying equality (3.6), we get

$$I = \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sup_{\|\boldsymbol{\xi}\|_{l_\infty} \leq 1} \sum_{\boldsymbol{h} \in G(\mathbf{M}_j)} \left| \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} \sum_{\boldsymbol{z} \in \mathbb{Z}^2} \widehat{f^{\boldsymbol{\xi}}}(\boldsymbol{z}) e^{i \boldsymbol{z}^T \left(\frac{\pi}{2} \mathbf{M}_j^{-T} \boldsymbol{k} \right)} e^{-i(4\boldsymbol{h})^T \left(\frac{\pi}{2} \mathbf{M}_j^{-T} \boldsymbol{k} \right)} \right|$$

$$= \frac{1}{\sqrt{|\det \mathbf{M}_j|}} \sup_{\|\boldsymbol{\xi}\|_{l_\infty} \leq 1} \sum_{\boldsymbol{h} \in G(\mathbf{M}_j)} \left| \sum_{\boldsymbol{k} \in \Omega_j \cap \mathbb{Z}^2} \sum_{\boldsymbol{z} \in \mathbb{Z}^2} \widehat{f^{\boldsymbol{\xi}}}(\boldsymbol{z}) e^{\frac{\pi}{2} i \boldsymbol{k}^T \left(\mathbf{M}_j^{-1} (\boldsymbol{z} - 4\boldsymbol{h}) \right)} \right|$$

$$= 16 \sqrt{|\det \mathbf{M}_j|} \sup_{\|\boldsymbol{\xi}\|_{l_\infty} \leq 1} \sum_{\boldsymbol{h} \in G(\mathbf{M}_j)} \left| \sum_{\boldsymbol{s} \in \mathbb{Z}^2} \widehat{f^{\boldsymbol{\xi}}} \left(4(\boldsymbol{h} + \mathbf{M}_j \boldsymbol{s}) \right) \right| \leq 16 \sqrt{|\det \mathbf{M}_j|} \sup_{\|\boldsymbol{\xi}\|_{l_\infty} \leq 1} \sum_{\boldsymbol{l} \in \mathbb{Z}^2} |\widehat{f^{\boldsymbol{\xi}}}(4\boldsymbol{l})|$$

$$= 16\sqrt{|\det \mathbf{M}_j|} \sup_{\|\boldsymbol{\xi}\|_{l_\infty} \le 1} \sum_{\boldsymbol{l} \in \mathbb{Z}^2} \left| \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} g_\alpha(\boldsymbol{x})\, e^{-4i(\boldsymbol{l}-\boldsymbol{\xi})^{\mathrm{T}}\boldsymbol{x}} d\boldsymbol{x} \right|$$

$$= 16\sqrt{|\det \mathbf{M}_j|} \sup_{\|\boldsymbol{\xi}\|_{l_\infty} \le 1} \sum_{\boldsymbol{l} \in \mathbb{Z}^2} |\widehat{g_\alpha}(4(\boldsymbol{l}-\boldsymbol{\xi}))|.$$

In view of Corollary 3.1, we have $g_\alpha \in C^{(2,2)}(\mathbb{T}^2)$. Besides, g_α satisfies condition (3.2) with $\boldsymbol{r} = (2,2)$. Consequently, applying (3.1) for all $\boldsymbol{k} \in \mathbb{R}^2$ and $\boldsymbol{r} = (2,2)$, we obtain

$$I \le 16\sqrt{|\det \mathbf{M}_j|} \left(\|g_\alpha\|_\infty + 2\sum_{l_1=1}^\infty \frac{1}{|l_1|^2} \left\| \frac{\partial^2 g_\alpha}{\partial x_1^2} \right\|_\infty + 2\sum_{l_2=1}^\infty \frac{1}{|l_2|^2} \left\| \frac{\partial^2 g_\alpha}{\partial x_2^2} \right\|_\infty \right.$$

$$\left. + 4\sum_{\boldsymbol{l} \in \mathbb{N}^2} \frac{1}{|l_1 l_2|^2} \left\| \frac{\partial^4 g_\alpha}{\partial x_1^2 \partial x_2^2} \right\|_\infty \right) \le C_1(\alpha)\sqrt{|\det \mathbf{M}_j|}, \quad C_1(\alpha) > 0.$$

$\square$

Remark 3.2. By a similar technique we can prove an analogous result for the functions $\psi_{\alpha,j}^\perp$, i.e.

$$(3.11) \qquad \left\| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |\psi_{\alpha,j}^\perp(\cdot - 2\pi\boldsymbol{y})| \right\|_\infty \le C_2(\alpha)\sqrt{|\det \mathbf{M}_j|}, \quad C_2(\alpha) > 0.$$

Now, we formulate and prove the main result of this section.

Theorem 3.1. *The orthogonal projection operator S_μ in (2.3) for the functions $\{t_k \colon k \in \mathbb{N}\}$ from Definition 2.3, acting as an operator from $C(\mathbb{T}^2)$ to $C(\mathbb{T}^2)$, is uniformly bounded for all $\mu \in \mathbb{N}$, i.e.*

$$\|S_\mu\|_{C(\mathbb{T}^2) \to C(\mathbb{T}^2)} \le C(\varepsilon), \quad C(\varepsilon) > 0.$$

Proof. Depending on the values of the parameter μ, we distinguish two cases.

Case I. Let $\mu = 2^j$, $j \in \mathbb{Z}_+$. Since the range V_j of the orthogonal projection operator S_μ is independent of the choice of its basis, we can choose the basis $\left\{ \mathrm{T}_{\boldsymbol{y}} \varphi_{\lambda,j}^\perp \colon \boldsymbol{y} \in P(\mathbf{M}_j) \right\}$ of V_j. Then, we have

$$\|S_{2^j}\|_{C(\mathbb{T}^2) \to C(\mathbb{T}^2)} = \sup_{\|f\|_\infty = 1} \sup_{\boldsymbol{x} \in \mathbb{T}^2} \left| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} \left\langle f, \varphi_{\lambda,j}^\perp(\cdot - 2\pi\boldsymbol{y}) \right\rangle \varphi_{\lambda,j}^\perp(\boldsymbol{x} - 2\pi\boldsymbol{y}) \right|$$

$$= \sup_{\|f\|_\infty = 1} \sup_{\boldsymbol{x} \in \mathbb{T}^2} \left| \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(\boldsymbol{\xi}) \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} \varphi_{\lambda,j}^\perp(\boldsymbol{x} - 2\pi\boldsymbol{y}) \overline{\varphi_{\lambda,j}^\perp(\boldsymbol{\xi} - 2\pi\boldsymbol{y})} \, d\boldsymbol{\xi} \right|$$

$$= \sup_{\boldsymbol{x} \in \mathbb{T}^2} \left\| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} \varphi_{\lambda,j}^\perp(\boldsymbol{x} - 2\pi\boldsymbol{y}) \overline{\varphi_{\lambda,j}^\perp(\cdot - 2\pi\boldsymbol{y})} \right\|_1$$

$$\le \sup_{\boldsymbol{x} \in \mathbb{T}^2} \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} \left| \varphi_{\lambda,j}^\perp(\boldsymbol{x} - 2\pi\boldsymbol{y}) \right| \left\| \varphi_{\lambda,j}^\perp(\cdot - 2\pi\boldsymbol{y}) \right\|_1$$

$$= \left\| \varphi_{\lambda,j}^\perp \right\|_1 \left\| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |\varphi_{\lambda,j}^\perp(\cdot - 2\pi\boldsymbol{y})| \right\|_\infty.$$

Now, using inequalities (3.4) and (3.10), we get

$$\|S_{2^j}\|_{C(\mathbb{T}^2) \to C(\mathbb{T}^2)} = \left\| \varphi_{\lambda,j}^\perp \right\|_1 \left\| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |\varphi_{\lambda,j}^\perp(\cdot - 2\pi\boldsymbol{y})| \right\|_\infty$$

$$(3.12) \qquad \leq \frac{1}{|\det \mathbf{M}_j|} \left\| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |\varphi_{\lambda,j}^\perp(\cdot - 2\pi\boldsymbol{y})| \right\|_\infty^2 \leq \frac{C_1^2(\lambda)}{|\det \mathbf{M}_j|} |\det \mathbf{M}_j| = C_1^2(\lambda).$$

Case II. Let now $2^j < \mu < 2^{j+1}$, $j \in \mathbb{Z}_+$. We decompose the orthogonal projection S_μ into the operator S_{2^j} and a remainder

$$S_\mu f = S_{2^j} f + \sum_{k=2^j+1}^\mu \langle f, t_k \rangle \, t_k.$$

Further, estimating the norm of the remainder operator, we obtain

$$\left\| \sum_{k=2^j+1}^\mu \langle f, t_k \rangle \, t_k \right\|_{C(\mathbb{T}^2) \to C(\mathbb{T}^2)} = \sup_{\|f\|_\infty = 1} \sup_{\boldsymbol{x} \in \mathbb{T}^2} \left| \sum_{k=2^j+1}^\mu \langle f, t_k \rangle \, t_k(\boldsymbol{x}) \right|$$

$$= \sup_{\|f\|_\infty = 1} \sup_{\boldsymbol{x} \in \mathbb{T}^2} \left| \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(\boldsymbol{\xi}) \sum_{k=2^j+1}^\mu t_k(\boldsymbol{x}) \overline{t_k(\boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{\xi} \right|$$

$$= \sup_{\boldsymbol{x} \in \mathbb{T}^2} \left\| \sum_{k=2^j+1}^\mu t_k(\boldsymbol{x}) \overline{t_k(\cdot)} \right\|_1 \leq \sup_{\boldsymbol{x} \in \mathbb{T}^2} \sum_{k=2^j+1}^{2^{j+1}} |t_k(\boldsymbol{x})| \, \|t_k\|_1$$

$$= \|\psi_{\lambda,j}^\perp\|_1 \left\| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |\psi_{\lambda,j}^\perp(\cdot - 2\pi\boldsymbol{y})| \right\|_\infty.$$

Now, using inequalities (3.4) and (3.11), we have

$$\left\| \sum_{k=2^j+1}^\mu \langle f, t_k \rangle \, t_k \right\|_{C(\mathbb{T}^2) \to C(\mathbb{T}^2)} = \|\psi_{\lambda,j}^\perp\|_1 \left\| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |\psi_{\lambda,j}^\perp(\cdot - 2\pi\boldsymbol{y})| \right\|_\infty$$

$$\leq \frac{1}{|\det \mathbf{M}_j|} \left\| \sum_{\boldsymbol{y} \in P(\mathbf{M}_j)} |\psi_{\lambda,j}^\perp(\cdot - 2\pi\boldsymbol{y})| \right\|_\infty^2$$

$$(3.13) \qquad \leq \frac{C_2^2(\lambda)}{|\det \mathbf{M}_j|} |\det \mathbf{M}_j| = C_2^2(\lambda).$$

Thus, in view of (3.12) and (3.13), we finally get for all $\mu \in \mathbb{N}$

$$\|S_\mu\|_{C(\mathbb{T}^2) \to C(\mathbb{T}^2)} \leq \|S_{2^j}\|_{C(\mathbb{T}^2) \to C(\mathbb{T}^2)} + \left\| \sum_{k=2^j+1}^\mu \langle f, t_k \rangle \, t_k \right\|_{C(\mathbb{T}^2) \to C(\mathbb{T}^2)}$$

$$\leq C_1^2(\lambda) + C_2^2(\lambda) = C(\varepsilon),$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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