Real function reconstruction from sparse Fourier samples

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In this paper we use the Prony method to reconstruct structured, real-valued functions from the smallest possible number of equidistantly distributed Fourier samples. In particular, we consider characteristic functions in $\mathbb{R}^2$ whose supports are unit-height polygons in the plane with N vertices. We show that these functions can be recovered by $3N$ Fourier samples.

1 Introduction

We want to show how the reconstruction of some structured, real-valued functions from a smallest possible set of Fourier data can be achieved by using the Prony method. In this paper, we will examine the case of unit-height polygons in the plane, i.e. characteristic functions of a polygonal domain in $\mathbb{R}^2$.

The Prony method is a method for parameter estimation for exponential sums, see e.g. [4], and is equivalent to the annihilating filter method, [7]. Consider a function $P(\omega)$ of the special form

$$P(\omega) = \sum_{j=1}^{N} c_j \mathrm{e}^{-\mathrm{i}\omega f_j}, \tag{1}$$

where the coefficients c_j and the frequencies f_j $(j = 1, \ldots, N)$ are real numbers with $c_j \neq 0$ and $f_1 < f_2 < \ldots < f_N$.

If the number N of summands in (1) is known, we can uniquely determine the unknown parameters c_j and f_j from $N+1$ function values $P(\ell h)$, $\ell = 0, \ldots, N$, where h is a positive constant such that $hf_j \in [-\pi, \pi)$ for all $j \in \{1, \ldots, N\}$. The Prony method also works if the number N is not known. Then, one needs an upper bound $M \geq N$ and $M+1$ function values $P(\ell h)$, $\ell = 0, \ldots, M$ and finds N by computing the rank of the Toeplitz matrix $\mathbf{T}_{M+1} := (P(h(\ell - m)))_{m,\ell=0}^{M}$, see [4–6]. If the sampling values are perturbed, the stability of the method has to be improved, see e.g. [2].

In [5], the reconstruction of structured, real-valued functions from a smallest possible set of Fourier samples using Prony's method has already been considered for some other classes of functions. In the univariate case the Prony method has been successfully applied to the reconstruction of step functions, non-uniform spline functions or non-uniform translates of a low-pass filter function. For the case of bivariate functions we can similarly reconstruct tensor products of non-uniform spline functions or non-uniform translates of radial functions, where the approach for the latter class of functions is also applicable to d-variate functions with $d > 2$, see [5].

The above mentioned function classes have in common that the Fourier data can be represented by exponential sums, i.e. by linear combinations of exponential terms, see [5]. In order to reconstruct the original functions, we can therefore exploit this special structure of the Fourier data by using the Prony method.

In [1], the authors proposed a method for reconstruction of polygonal shapes from moments. Now, in the present paper, we show how unit-height polygons in the plane can be reconstructed from sparse Fourier samples using Prony's method.

2 Reconstruction of polygonal shapes

Let us consider a function f of the special form

$$f(x) = \mathbf{1}_D(x), \quad x \in \mathbb{R}^2, \tag{2}$$

where $\mathbf{1}_D$ is the characteristic function of the domain $D \subset \mathbb{R}^2$. Here, D is a polygonal domain with the N vertices $v_j \in \mathbb{R}^2$, $j = 1, \ldots, N$.

We number the vertices of the polygon anticlockwise and use the notation $\widetilde{\eta}_j$ for the perpendicular vector to the edge $v_{j+1} - v_j$ of the polygon which is pointing outward. Further, we use the convention

$$v_{N+1} := v_1, \quad v_0 := v_N, \quad \widetilde{\eta}_0 := \widetilde{\eta}_N.$$

Then we get

$$\widehat{f}(\xi) = \frac{1}{\|\xi\|_2^2} \sum_{j=1}^{N} \left(\frac{\widetilde{\eta}_j^T \xi}{(v_{j+1} - v_j)^T \xi} - \frac{\widetilde{\eta}_{j-1}^T \xi}{(v_j - v_{j-1})^T \xi} \right) \mathrm{e}^{-\mathrm{i}\xi^T v_j} \tag{3}$$

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as the representation for the Fourier transform of f if $\xi \neq \mathbf{0}$ and $(v_{j+1} - v_j)^T \xi \neq 0$ for all $j \in \{0, \dots, N\}$, see [3], where a^T denotes the transpose of $a \in \mathbb{R}^2$.

Hence, the Fourier data can be expressed in the form of an exponential sum and we can use this structure to determine the parameters of the original function f (i.e. the vertices and the order of the vertices) from sparse Fourier samples by applying the Prony method. Let us assume that no edge of the polygon D is parallel to the x-axis or the y-axis in the plane. Then we have

$$(v_{j+1} - v_j)^T \xi \neq 0 \quad \text{for all } j \in \{0, \dots, N\}$$

for vectors ξ of the form $\xi = (\xi_1, 0)^T$ or $\xi = (0, \xi_2)^T$ with $\xi_1, \xi_2 \neq 0$, and the formula in (3) can be applied.

Let $v_j := (\alpha_j, \beta_j)^T$ for all $j \in \{1, \dots, N\}$. With the further assumption that the components $\alpha_j, j = 1, \dots, N$, are pairwise different and that the components $\beta_j, j = 1, \dots, N$, are pairwise different we can prove the following theorem by extending the approach in [5].

Theorem 2.1 *Let the before mentioned assumptions on the vertices v_j of the polygon D be satisfied, and let $h > 0$ such that $h\|v_j\|_2 < \pi$ for all $j \in \{1, \dots, N\}$. Then the polygon D can be uniquely recovered from the $3N$ Fourier samples $\widehat{f}(\xi)$ with*

$$\xi \in \{(\ell h, 0), (0, \ell h), (\cos(\varphi\pi)\ell h, \sin(\varphi\pi)\ell h)\}, \quad \ell = 1, \dots, N,$$

where $\varphi \in (0,1) \setminus \{\frac{1}{2}\}$ needs to be chosen suitably.

The idea of the proof is as follows. First, we compute the vertices v_j of D using an approach similar to [5, Section 4.2], i.e., we apply the Prony method to sampling values from three straight lines through the origin, where the parameter φ is determined adaptively. Afterwards, these vertices have to be ordered, since the polygon D can be concave. But by applying the Prony method in order to compute the vertices, we also obtain coefficients for a representation of $\widehat{f}$ similar to (3). We can use these coefficients to order the computed vertices.

We want to show the applicability of this approach with an example. Figure 1 presents a unit-height polygon in the plane with the five vertices given in Table 1 (i.e. $N = 5$). Observe that the difference of the first components of the second and fifth vertex is rather small, as well as the difference of α_1 and α_3 and the difference of β_1 and β_4. For the reconstruction process 15 sampling values $\widehat{f}(\xi)$ with sampling locations ξ according to Theorem 2.1 are considered, where the stepsize h is chosen as $h = 0.4$. Further, we use $\varphi = 0.25$. Table 1 also presents the absolute reconstruction errors $|\alpha_j - \alpha_j^*|$ and $|\beta_j - \beta_j^*|$ ($j = 1, \dots, 5$), where α_j^* and β_j^* denote the values for the reconstructed vertex components.

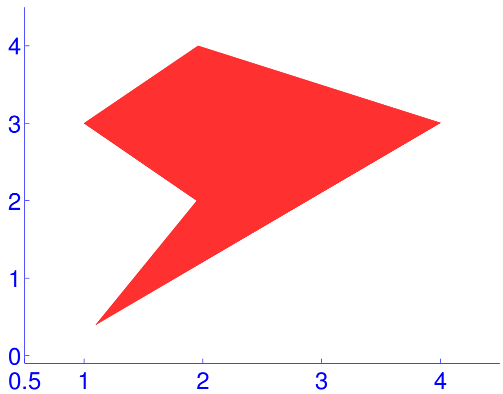

Fig. 1: Original function of the form (2) determined by vertices v_j given in Table 1.

Table 1: Vertices $v_j = (\alpha_j, \beta_j)^T$ of the polygon given in Figure 1 and reconstruction errors, where $h = 0.4$ and $\varphi = 0.25$.

j	α_j	$\lvert\alpha_j^* - \alpha_j\rvert$	β_j	$\lvert\beta_j^* - \beta_j\rvert$
1	1	$1.601 \cdot 10^{-9}$	3	$2.792 \cdot 10^{-7}$
2	1.95	$2.677 \cdot 10^{-7}$	2	$5.788 \cdot 10^{-12}$
3	1.1	$5.524 \cdot 10^{-9}$	0.4	$1.197 \cdot 10^{-13}$
4	4	$1.403 \cdot 10^{-13}$	3.005	$1.68 \cdot 10^{-7}$
5	1.96	$4.961 \cdot 10^{-7}$	4	$7.994 \cdot 10^{-13}$

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