Prony's Method for Multivariate Signals

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The problem of recovering translates and corresponding amplitudes of sparse sums of Gaussians out of sampling values as well as reconstructing sparse sums of exponentials are nonlinear inverse problems that can be solved for example by Prony's method. Here, we want to demonstrate a new extension to multivariate input data.

1 Introduction

Assume we only know equidistant measurements $f(k)$, $k = 0, \ldots, N$, of the signal $f(x) = \sum_{j=1}^{M} c_j \mathrm{e}^{t_j x}$, $t_j, c_j \in \mathbf{C}$. The task is to recover the parameters t_j and the corresponding coefficients c_j. Methods that accomplish this problem include super-resolution [1], Prony's Method [4], ESPRIT [9], MUSIC [10], Matrix-Pencil-Method [2], or the Annihilating Filter Method for signals with finite rate of innovation [11], where all but the first method can be seen as Prony-like methods [7].

In [1], it is demonstrated that minimizing total variation is also applicable for multivariate signals. Another way to analyze multivariate signals is to reduce the problem to a number of one-dimensional problems via projections of the data to multiple lines through the origin and to apply a Prony-like method to each projection. Recent recovering results in [6, 8] using this approach work only if the coefficients have equal sign, i.e., $c_j \in \mathbf{R}_+$, since the inherited projection might otherwise cause cancellation. In contrast, we established a new, fully multidimensional Prony method that is applicable for arbitrary $c_j \in \mathbf{C}$. This advantage comes at the price of calculating common zeros of multivariate polynomials, which is a challenging task itself.

2 Reconstructing Multivariate Exponentials

In this section we present a direct generalization of Prony's method to d dimensions. Note, that setting $d = 1$ breaks down the upcoming calculations to the standard Prony method. Instead of the signal $f(x)$, as introduced above, we now consider an M-sparse sum of d-variate exponentials, $f(x) = \sum_{j=1}^{M} c_j \mathrm{e}^{\langle x, t_j \rangle}$, $x, t_j \in \mathbf{C}^d$, $c_j \in \mathbf{C}$. We define d-variate Prony polynomials $P : \mathbf{C}^d \to \mathbf{C}$, $P(z) := \sum_{k=0}^{N} p_k z^{n_k}$, $n_k \in \mathbf{N}^d$, with sufficiently large $N \geq M$ such that $P(\mathrm{e}^{t_j}) = 0$ for $j = 1, \ldots, M$ where $P(\mathrm{e}^{t_j}) := \sum_{k=0}^{N} p_k \prod_{\ell=1}^{d} (\mathrm{e}^{t_{j,\ell}})^{n_{k,\ell}} = \sum_{k=0}^{N} p_k \mathrm{e}^{\langle n_k, t_j \rangle}$. For arbitrary shifts $m_\ell \in \mathbf{N}^d$ we observe

$$\sum_{k=0}^{N} p_k f(n_k + m_\ell) = \sum_{j=1}^{M} c_j \mathrm{e}^{\langle m_\ell, t_j \rangle} \sum_{k=0}^{N} p_k \mathrm{e}^{\langle n_k, t_j \rangle} = \sum_{j=1}^{M} c_j \mathrm{e}^{\langle m_\ell, t_j \rangle} P(\mathrm{e}^{t_j}) = 0. \tag{1}$$

Thus, we have to solve the linear problem $Hp = (f(n_k + m_\ell))_{k,\ell=0}^{N} (p_k)_{k=0}^{N} = 0$ in order to find the coefficients p_k of the Prony polynomials $P(z)$. The roots e^{t_j}, $j = 1, \ldots, M$, of these polynomials carry the information of the unknown parameters t_j. Note that in one dimension we just have to compute the zeros of one Prony polynomial $P(z)$ of order $N = M$ to find the parameters e^{t_j}. Polynomials in d variables on the other hand have $d-1$-dimensional zero sets, so those sets are too large to extract the desired values e^{t_j}. But, by construction, the values e^{t_j} are contained in the zero set of every Prony polynomial whose coefficient-vector lies in the kernel of H. The idea is now to construct a matrix H, where the dimension of the kernel is large enough, such that we can ensure that the intersection of the zero sets of the obtained Prony polynomials is just the set $\Omega := \{\mathrm{e}^{t_j} \,|\, j = 1, \ldots, M\}$. In [3] it is shown that for suitably chosen sampling points n_k and shifts m_ℓ the kernel of dimension $(M+1)^d - M$ of H ensures that the zero sets of the Prony polynomials associated with $\ker(H)$ intersect only in Ω. Note that in the typical case a much smaller kernel of dimension $d-1$ suffices.

After extracting the parameters t_j out of the roots, we determine the coefficients c_j as least squares solution of the Vandermonde-type system $\left(\mathrm{e}^{\langle t_j, n_k \rangle}\right)_{k=0,j=1}^{N,M} (c_j)_{j=1}^{M} = (f(n_k))_{k=0}^{N}$.

3 Reconstructing Translations of Multivariate Gaussians

If the 1-dimensional Gaussian $K_{1,b}(x) := \mathrm{e}^{-bx^2}$, $b > 0$, is known beforehand, the 1-dimensional Prony method can also be used to recover translates t_j and corresponding coefficients c_j of a signal $s(x) = \sum_{j=1}^{M} c_j \mathrm{e}^{-b(x-t_j)^2} = \sum_{j=1}^{M} c_j K_{1,b}(x - t_j)$,

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$x \in \mathbf{R}, t_j \in [0,1], c_j \in \mathbf{C}$, from sufficiently many sampling values $s(k), k = 0, \dots, N$, if the data is transferred to the Fourier-domain [5]. Here, we want to demonstrate an algorithm for recovering multivariate translates $t_j \in [0,1]^d$ of the signal

$$s(x) = \sum_{j=1}^{M} c_j \mathrm{e}^{-b(x-t_j)^{\mathrm{T}}(x-t_j)} = \sum_{j=1}^{M} c_j K_{d,b}(x - t_j), \quad x \in \mathbf{R}^d, t_j \in [0,1]^d, c_j \in \mathbf{C},$$

directly in the spacial domain, with $K_{d,b}(x) := \mathrm{e}^{-bx^{\mathrm{T}}x}$. For the multivariate signal $s(x)$ we consider again multivariate Prony polynomials $P : \mathbf{C}^d \to \mathbf{C}$, $P(z) = \sum_{k=0}^{N} p_k z^{n_k}$, but now with roots e^{2bt_j}, i.e., $P(\mathrm{e}^{2bt_j}) = 0$, $j = 1, ..., M$. Let $\mathcal{N} = \mathcal{N}_P := \{n_k \,|\, k = 0, \dots, N\}$ be the set containing all exponents n_k of the multivariate monomials $z^{n_k} = z_1^{n_{k,1}} \cdots z_d^{n_{k,d}}$ that are active in $P(z)$. For $\alpha(m, n_k) := \mathrm{e}^{2bn_k^{\mathrm{T}}m}$ and $q_k := p_k \mathrm{e}^{bn_k^{\mathrm{T}}n_k} = K_{d,b}(n_k)^{-1} p_k$, with $n_k \in \mathcal{N} \subset \mathbf{N}^d$ and shifts $m \in \mathbf{N}^d$ we get

$$\sum_{k=0}^{N} q_k s(n_k + m)\alpha(m, n_k) = \sum_{j=1}^{M} c_j K_{d,b}(m - t_j) \sum_{k=0}^{N} p_k \mathrm{e}^{2bn_k^{\mathrm{T}}t_j} = \sum_{j=1}^{M} c_j K_{d,b}(m - t_j)\, P(\mathrm{e}^{2bt_j}) = 0$$

in analogy to the calculation in (1). Thus, we have to solve the linear system $Hq = 0$, $H = \left(s(n_k + m_\ell)\mathrm{e}^{2bm_\ell^{\mathrm{T}}n_k}\right)_{\ell,k=0}^{N}$ with $q = (p_k K_{d,b}(n_k)^{-1})_{k=0}^{N}$. Once, we have calculated the coefficients q_k, we can evaluate the coefficients p_k of the Prony polynomial $P(z)$. By construction, the translates t_j are contained in the $(d-1)$-dimensional zero set of $P(z)$. Again, we refer to [3] for a proof that $\dim(\ker(H)) = (M+1)^d - M$ suffices for unique reconstruction. After finding t_j, $j = 1, \dots, M$, the coefficients c_j can be determined as a least squares solution of $(K_{d,b}(n_k - t_j))_{k=0,j=1}^{N,M} (c_j)_{j=1}^{M} = (s(n_k))_{k=0}^{N}$.

Algorithm for multivariate exponentials
Input: $f(n_k + m_\ell), n_k, m_\ell, k, \ell = 0, \dots, N$

1. Calculate all vectors p in the kernel of $H = (f(m_\ell + n_k))_{\ell,k=0}^{N}$ and construct the polynomials $P(z) = \sum_{k=0}^{N} p_k z^{n_k}$.
2. Find the common zeros e^{t_j}, $j = 1, \dots, M$, of at least $d+1$ polynomials $P(z)$.
3. Find a least squares solution of the linear system $(\mathrm{e}^{\langle n_k, t_j\rangle})_{k=0,j=1}^{N,M} (c_j)_{j=1}^{M} = (f(n_k))_{k=0}^{N}$.

Output: M, t_j, c_j.

Algorithm for multivariate Gaussians
Input: $s(n_k + m_\ell), n_k, m_\ell, k, \ell = 0, \dots, N, b > 0$

1. Calculate all vectors q in the kernel of $H = (s(n_k + m_l)\mathrm{e}^{2bm_\ell^{\mathrm{T}}n_k})_{\ell,k=0}^{N}$ and construct the polynomials $P(z) = \sum_{k=0}^{N} (q_k \mathrm{e}^{-bn_k^{\mathrm{T}}n_k}) z^{n_k}$.
2. Find the common zeros t_j, $j = 1, \dots, M$, of at least $d+1$ polynomials $P(z)$.
3. Find a least squares solution of the linear system $(\mathrm{e}^{-b(n_k - t_j)^{\mathrm{T}}(n_k - t_j)})_{k=0,j=1}^{N,M} (c_j)_{j=1}^{M} = (s(n_k))_{k=0}^{N}$.

Output: M, t_j, c_j.

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