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Ambiguities in one-dimensional phase retrieval of structured functions

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Phase retrieval means that we wish to recover a complex-valued signal (discrete or continuous) from the magnitudes of its Fourier transform. Here we restrict ourselves to the recovery of structured functions, e.g. linear spline functions with equidistant knots. First, a complete characterization of the occurring ambiguities is presented. Moreover, we investigate additionally given moduli of the signal itself regarding their ability to reduce the set of ambiguities.

1 Introduction

The phase retrieval problem appears in a wide range of applications in physics, e.g. in crystallography, electron microscopy, astronomy, and optics. Here we wish to recover an unknown structured function f of the form

$$f(t) := \sum_{n \in \mathbf{Z}} c[n]\, \phi(t-n) \tag{1}$$

from its Fourier intensities $|\widehat{f}\,|$ given by

$$\left|\widehat{f}(\omega)\right| = \left|\int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t\right|, \tag{2}$$

where ϕ is an a priori given generator function in $C^0 \cap L^1$ not equal to zero, and where $(c[n])_{n\in\mathbf{Z}}$ is the complex-valued sequence with finite support. If we choose the centred linear B-spline $\phi(t) := (1 - |\,t\,|)\, \chi_{[-1,1]}(t)$ as generator function, then the unknown function f is a linear spline function. This specific phase retrieval problem was introduced in [1, 2]. Obviously, the considered phase retrieval problem is not uniquely solvable. For example, we can construct further solutions simply by rotating (multiplying a unimodular constant), shifting, or reflecting and conjugating a given solution. Moreover, besides these trivial ambiguities, there exist further non-trivial ambiguities.

2 Phase retrieval of discrete-time signals

In order to characterize all ambiguities – trivial and non-trivial – of the phase retrieval problem to recover a structured function, we can use our findings in [3], where we consider the problem to reconstruct an unknown complex-valued signal $x := (x[n])_{n\in\mathbf{Z}}$ with finite support from its discrete-time Fourier intensities $|\,\widehat{x}\,|$ given by $|\,\widehat{x}(\omega)\,| = |\sum_{n\in\mathbf{Z}} x[n]\, \mathrm{e}^{-\mathrm{i}\omega n}\,|$. Denoting the support length of the signal x by N, we introduce the autocorrelation signal and the corresponding autocorrelation polynomial defined by

$$a[n] := \sum_{k\in\mathbf{Z}} x[k]\, \overline{x[k+n]} \qquad \text{and} \qquad P(z) := z^{N-1} \sum_{n=-N+1}^{N-1} a[n]\, z^n \tag{3}$$

respectively, where $(a[n])_{n\in\mathbf{Z}}$ is known from $|\,\widehat{x}(\omega)\,|^2 = \sum_{n\in\mathbf{Z}} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n}$. Now, we can characterize all ambiguities in the following manner, cf. [3, Theorem 2.4].

Theorem 2.1 *Let x be a discrete-time signal with finite support length N. Then the Fourier transform of each discrete-time signal y satisfying $|\,\widehat{y}\,| = |\,\widehat{x}\,|$ can be written in the form*

$$\widehat{y}(\omega) = \mathrm{e}^{\mathrm{i}(\alpha+\omega n_0)} \sqrt{|\,a[N-1]\,| \prod_{j=1}^{N-1} |\,\beta_j\,|^{-1}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right)$$

where $\alpha \in [-\pi, \pi)$, $n_0 \in \mathbf{Z}$, and β_j is chosen from the zero pairs $(\gamma_j, \overline{\gamma}_j^{-1})$ of the autocorrelation polynomial P.

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There have been many different approaches, such as additional interference measurements [4, 5], to enforce uniqueness of the solution or to reduce the set of non-trivial solutions of the one-dimensional phase retrieval problem for real-valued discrete-time signals. Using Theorem 2.1, we can now examine the quality of these approaches also for the recovery of complex-valued discrete-time signals, see [3]. In some applications, besides the moduli of the Fourier transform, the moduli of the unknown complex-valued signal itself are known. With these additional constraints, the set of non-trivial ambiguities is almost always reduced to one unique solution up to trivial ambiguities caused by rotations, see [3, Section 3.4]. More precisely, we have the following result.

Theorem 2.2 *Almost every signal x can be uniquely recovered from its Fourier intensities $|\widehat{x}|$ and the moduli $|x|$ of the signal itself up to rotations.*

3 Phase retrieval of structured functions

Now, we want to apply the result in Theorem 2.1 on discrete-time signals in order to characterize all ambiguities in the recovery of a structured function f of the form (1) from its Fourier intensities.

Corollary 3.1 *Let f be a structured function* (1) *with finite support length N of the corresponding coefficients sequence $(c[n])_{n\in\mathbf{Z}}$. Then the Fourier transform of each structured function g satisfying $|\widehat{g}| = |\widehat{f}|$ can be written in the form*

$$\widehat{g}(\omega) = \mathrm{e}^{\mathrm{i}(\alpha+\omega n_0)}\,\widehat{\phi}(\omega)\sqrt{|a[N-1]|\prod_{j=1}^{N-1}|\beta_j|^{-1}\cdot\prod_{j=1}^{N-1}\left(\mathrm{e}^{-\mathrm{i}\omega}-\beta_j\right)}$$

where $\alpha \in [-\pi, \pi)$, $n_0 \in \mathbf{Z}$, and β_j is chosen from the zero pairs $(\gamma_j, \overline{\gamma}_j^{-1})$ of the autocorrelation polynomial P in (3) *with the autocorrelation sequence $(a[n])_{n\in\mathbf{Z}}$ determined by $\sum_{n\in\mathbf{Z}} a[n]\,\mathrm{e}^{-\mathrm{i}\omega n} = |\widehat{f}(\omega)/\widehat{\phi}(\omega)|^2$ whenever $\widehat{\phi}(\omega) \neq 0$.*

Proof. Let g defined by $g(t) := \sum_{n\in\mathbf{Z}} b[n]\,\phi(t-n)$ be a structured function with the same Fourier intensities as f. Hence, the squared Fourier intensities of the structured function f and g have to fulfil the equation

$$\left|\widehat{f}(\omega)\right|^2 = \left|\widehat{\phi}(\omega)\right|^2\left|\sum_{n\in\mathbf{Z}} c[n]\,\mathrm{e}^{-\mathrm{i}\omega n}\right|^2 = \left|\widehat{\phi}(\omega)\right|^2\left|\sum_{n\in\mathbf{Z}} b[n]\,\mathrm{e}^{-\mathrm{i}\omega n}\right|^2 = |\widehat{g}(\omega)|^2 .$$

Since the squared discrete-time Fourier intensities of the coefficients sequences $c := (c[n])_{n\in\mathbf{Z}}$ and $b := (b[n])_{n\in\mathbf{Z}}$ are non-negative trigonometric polynomials (the sequences have only a finite support), and since $\widehat{\phi}$ is continuous and not zero everywhere, the squared Fourier intensities of the signals c and b can be completely determined by the intensities $|\widehat{f}|$ or $|\widehat{g}|$.

Therefore, the phase retrieval problem for the structured function f can be reduced to the phase retrieval problem to recover the coefficients sequence $(c[n])_{n\in\mathbf{Z}}$ from the corresponding discrete-time Fourier intensities. Thus, the occurring ambiguities in the recovery of the coefficients sequence with finite support are characterized by Theorem 2.1. □

Finally, we are interested in reducing the ambiguities of the phase retrieval problem for structured functions. Because of the close relation to the phase retrieval problem for discrete-time signals, the ideas in [3] to employ additional interference measurements for ambiguity reduction can be simply transferred. For Theorem 2.2, however, we need further appropriate side conditions. More precisely, we assume that the generator function ϕ is a Lagrange function, i.e. $\phi(n) = \delta(n)$ for all $n \in \mathbf{Z}$, cf. [6].

Corollary 3.2 *Let ϕ be a Lagrange function. Then almost every structured function f can be uniquely recovered from its Fourier intensities $|\widehat{f}|$ and the moduli $|f(n)|$ for $n \in \mathbf{Z}$ up to rotations.*

Proof. With the identity $|f(n)| = |c[n]|$, the considered problem can be reduced to the recovery of the coefficients sequence $c = (c[n])_{n\in\mathbf{Z}}$ from the Fourier intensities $|\widehat{c}|$ and the moduli $|c|$. Now, the application of Theorem 2.2 yields the assumption. □

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