APPROXIMATION OF THE CLASSES H_p^Ω OF PERIODIC FUNCTIONS OF MANY VARIABLES IN THE SPACE L_p

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We establish upper estimates for the approximation of the classes H_p^Ω of periodic functions of many variables by polynomials constructed by using the system obtained as the tensor product of the systems of functions of one variable. These results are then used to establish the exact-order estimates of the orthoprojective widths for the classes H_p^Ω in the space L_p with $p \in \{1, \infty\}$.

1. Introduction

In the present paper, we study the problems of approximation of periodic functions of many variables from the classes H_p^Ω by polynomials constructed by using the system of functions obtained as the tensor product of systems of functions of one variable. The trigonometric system $\{e^{i(\mathrm{k},\mathrm{x})}\}_{\mathrm{k}\in\mathbb{Z}^d}$ is a classical example of a system of this kind:

$$e^{i(\mathrm{k},\mathrm{x})} = \prod_{j=1}^{d} e^{ik_j x_j}, \quad \mathrm{x} = (x_1, \ldots, x_d),$$

where $\mathbb{Z}^d$ is an integer-valued d-dimensional lattice.

The Haar system $\{H_I(\mathrm{x})\}$:

$$H_I(\mathrm{x}) = \prod_{j=1}^{d} H_{I_j}(x_j), \qquad I = I_1 \times \ldots \times I_d, \quad \mathrm{x} = (x_1, \ldots, x_d),$$

where I_j stands for the double integral, which is the support of the Haar function $H_{I_j}(t)$, $t \in \mathbb{R}$, is another important example.

For a more detailed statement of the problem, we present necessary notation and definitions.

Let $\mathbb{R}^d$, $d \geq 1$, be a d-dimensional Euclidean space with elements $\mathrm{x} = (x_1, \ldots, x_d)$, $\mathrm{y} = (y_1, \ldots, y_d)$, $(\mathrm{x}, \mathrm{y}) = x_1 y_1 + \ldots + x_d y_d$, and let $L_p(\pi_d)$, $\pi_d = \prod_{j=1}^{d} [0, 2\pi]$, be a space of functions $f(\mathrm{x}) = f(x_1, \ldots, x_d)$ 2π-periodic in each variable and summable to the power p, $1 \leq p < \infty$ (and essentially bounded for $p = \infty$) in the cube π_d whose norm is defined as follows:

$$\|f\|_{L_p(\pi_d)} = \|f\|_p = \left((2\pi)^{-d} \int_{\pi_d} |f(\mathrm{x})|^p d\mathrm{x} \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L_\infty(\pi_d)} = \|f\|_\infty = \operatorname*{ess\,sup}_{\mathrm{x}\in\pi_d} |f(\mathrm{x})|.$$

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In the present paper, we consider only functions $f \in L_p(\pi_d)$ satisfying the condition

$$\int_0^{2\pi} f(\mathrm{x}) dx_j = 0, \quad j = \overline{1, d}.$$

For simplicity, instead of $L_p(\pi_d)$, we write L_p.

For $f \in L_p$ and $\mathrm{h} \in \mathbb{R}^d$, we define a mixed difference of order l by the formula

$$\Delta^l_{\mathrm{h}} f(\mathrm{x}) = \Delta^l_{h_d}(\ldots(\Delta^l_{h_1} f(\mathrm{x}))\ldots),$$

where

$$\Delta^l_{h_j} f(\mathrm{x}) = \sum_{n=0}^{l} (-1)^{l-n} C_l^n f(x_1, \ldots, x_{j-1}, x_j + nh_j, x_{j+1}, \ldots, x_d).$$

For $f \in L_p$ and $\mathrm{t} = (t_1, \ldots, t_d),\ t_j \geq 0,\ j = \overline{1, d}$, we define a mixed modulus of smoothness of order $l \in \mathbb{N}$ by the formula

$$\Omega_l(f, \mathrm{t})_p = \sup_{|h_j| \leq t_j, j = \overline{1,d}} \|\Delta^l_{\mathrm{h}} f(\cdot)\|_p.$$

Let $\Omega(\mathrm{t}) = \Omega(t_1, \ldots, t_d)$ be a function of the type of mixed modulus of smoothness of order l, i.e., a function defined on $\mathbb{R}^d_+ = \{\mathrm{t} \in \mathbb{R}^d \colon t_j \geq 0,\ j = \overline{1, d}\}$ and satisfying the following conditions:

(i) $\Omega(\mathrm{t}) > 0,\ t_j > 0,\ j = \overline{1, d}$, and $\Omega(\mathrm{t}) = 0,\ \prod_{j=1}^d t_j = 0$;

(ii) $\Omega(\mathrm{t})$ does not decrease in each variable $t_j \geq 0,\ j = \overline{1, d}$, for all values of the other variables $t_i,\ i \neq j$;

(iii) $\Omega(m_1 t_1, \ldots, m_d t_d) \leq \left(\prod_{j=1}^d m_j\right)^l \Omega(\mathrm{t}),\ m_j \in \mathbb{N},\ j = \overline{1, d}$;

(iv) $\Omega(\mathrm{t})$ is continuous for $t_j \geq 0,\ j = \overline{1, d}$.

The set of these functions Ω is denoted by Ψ_l.

For a given function $\Omega \in \Psi_l$, we define a class of functions (see, e.g., [1])

$$H^\Omega_p = \{f \in L_p \colon \Omega_l(f, \mathrm{t})_p \leq \Omega(\mathrm{t})\}.$$

Note that, in the case where $\mathrm{r} = (r_1, \ldots, r_d),\ 0 < r_j < l,\ j = \overline{1, d}$, and $\Omega(\mathrm{t}) = \prod_{j=1}^d t_j^{r_j}$, the classes H^Ω_p coincide with the well-known Nikol'skii classes H^{r}_p [2]. We also assume that Ω belongs to the sets S^α and S_l.

We say that a function of one variable φ belongs to $S^\alpha,\ \alpha > 0$, if the function $\varphi(\tau)/\tau^\alpha$ almost increases, i.e., there exists a constant $C_1 > 0$ independent of τ_1 and τ_2 such that

$$\frac{\varphi(\tau_1)}{\tau_1^\alpha} \leq C_1 \frac{\varphi(\tau_2)}{\tau_2^\alpha}, \quad 0 < \tau_1 \leq \tau_2.$$

A function φ belongs to S_l if there exists $\gamma,\ 0 < \gamma < l$, such that the function $\varphi(\tau)/\tau^\gamma$ is almost decreasing, i.e., there exists a constant $C_2 > 0$ independent of τ_1 and τ_2 such that

$$\frac{\varphi(\tau_1)}{\tau_1^\gamma} \geq C_2 \frac{\varphi(\tau_2)}{\tau_2^\gamma}, \quad 0 < \tau_1 \leq \tau_2.$$

The conditions under which a function belongs to the sets S^α and S_l are called the Bari–Stechkin conditions [3].

Assume that Ω belongs to S^α (respectively, Ω belongs to S_l) if $\Omega(t_1, \ldots, t_d)$, as a function of the variable t_j, $j = \overline{1, d}$, for all values of the other variables $t_i,\ i \neq j$, belongs to the set S^α (respectively, to the set S_l).

Denote $\Phi_{\alpha,l} = \Psi_l \cap S^\alpha \cap S_l$.

Further, assume that, for two nonnegative quantities A and B, the relation $A \asymp B$ means that there are constants $C_3,\ C_4 > 0$ such that $C_3 A \leq B \leq C_4 A$. The relations $A \ll B$ or $A \gg B$ mean that $C_5 A \leq B$ and $B \leq C_6 A,\ C_5, C_6 > 0$, respectively. The constants $C_i,\ i = 1, 2, \ldots,$ used in the present paper may depend only on the parameters from the definitions of a class and of a metric in which the accuracy of approximation is estimated, as well as on the dimension of the space $\mathbb{R}^d$.

By $V_m(x),\ m \in \mathbb{N},\ x \in \mathbb{R}$, we denote the de-la-Vallée-Poussin kernel

$$V_m(x) = 1 + 2\sum_{k=1}^{m} \cos kx + 2\sum_{k=m+1}^{2m-1} \left(\frac{2m-k}{m}\right) \cos kx\ .$$

For a function $f \in L_p$ and a vector $\mathrm{s} \in \mathbb{Z}_+^d$, we consider the polynomial

$$A_{\mathrm{s}}(f) = f * \prod_{j=1}^{d} (V_{2^{s_j-1}} - V_{2^{s_j-2}}).$$

The following theorem on functions belonging to the class H_p^Ω was proved in [1]:

Theorem A. *Let a function Ω belong to $\Phi_{\alpha,l},\ \alpha > 0$. Then f belongs to $H_p^\Omega,\ 1 \leq p \leq \infty$, if and only if the following order inequality is true:*

$$\|A_{\mathrm{s}}(f)\|_p \ll \Omega(2^{-\mathrm{s}}), \tag{1}$$

where $2^{-\mathrm{s}} = (2^{-s_1}, \ldots, 2^{-s_d})$.

For the construction of approximate aggregates, we now introduce some sets. For any $N \in \mathbb{N}$, we denote

$$\kappa(N) = \kappa(\Omega, N) = \left\{ \mathrm{s} = (s_1, \ldots, s_d) \colon s_j \in \mathbb{N}, \Omega(2^{-\mathrm{s}}) \geq \frac{1}{N} \right\}, \tag{2}$$

$$\kappa^\perp(N) = \left\{ \mathrm{s} = (s_1, \ldots, s_d) \colon s_j \in \mathbb{N}, \Omega(2^{-\mathrm{s}}) < \frac{1}{N} \right\}, \tag{3}$$

$$\Theta(N) = \kappa^\perp(N) \setminus \kappa^\perp(2^l N). \tag{4}$$

It follows from (3) and (4) that $\Theta(N) \subset \kappa^\perp(N)$ and $\Theta(N) \cap \kappa^\perp(2^l N) = \varnothing$, i.e.,

$$\frac{1}{2^l N} \leq \Omega(2^{-\mathrm{s}}) < \frac{1}{N}$$

or

$$\Omega(2^{-\mathrm{s}}) \asymp \frac{1}{N}, \quad \mathrm{s} \in \Theta(N). \tag{5}$$

In [4], it is shown that the following relation is true:

$$|\Theta(N)| \asymp (\log_2 N)^{d-1}, \tag{6}$$

where $|\mathcal{M}|$ is the number of elements in the set $\mathcal{M}$.

To prove our main results, we need the following lemma:

Lemma A [4]. *Let a function* Ω *belong to* $\Psi_l \cap S^\alpha,\ \alpha > 0$. *Then, for* $0 < p < \infty$,

$$\sum_{\mathrm{s}\in\kappa^{\perp}(N)} (\Omega(2^{-\mathrm{s}}))^p \ll \sum_{\mathrm{s}\in\Theta(N)} (\Omega(2^{-\mathrm{s}}))^p.$$

We define the operator $\mathbf{F}_\rho$ as an operator of convolution with the Bernoulli kernel,

$$F_\rho(x) = 1 + 2\sum_{k=1}^{\infty} k^{-\rho} \cos\left(kx - \frac{\rho\pi}{2}\right), \qquad x \in \mathbb{R}, \quad \rho > 0.$$

By $\mathbf{F}_\rho(L_p)$, we denote a set of functions defined in the form of the convolution of the Bernoulli kernel with some function $\varphi \in L_p$, i.e.,

$$\mathbf{F}_\rho(L_p) = \left\{ f \in L_p \colon f = \varphi * F_\rho,\ \varphi \in L_p \right\}.$$

Further, we define the classes of Sobolev functions W_p^ρ considered in what follows,

$$W_p^\rho = \left\{ f \colon f(x) = \frac{1}{2\pi} \int\limits_{\pi_1} F_\rho(x-y)\varphi(y)dy,\ \varphi \in L_p, \|\varphi\|_p \le 1 \right\}.$$

Consider a set of operators $\{Y_n\}_{n=0}^{\infty}$ defined on $\mathbf{F}_\rho(L_p)$ with the properties:

(A) $\|(I - Y_n)\mathbf{F}_\rho\|_{p\to p} \ll 2^{-\rho n},\ n \in \mathbb{Z}_+$, where I is the identity operator and $\|T\|_{p\to p} = \|T\|_{L_p \to L_p}$ is the norm of the operator T from L_p into L_p;

(B) for an arbitrary trigonometric polynomial t of degree 2^μ and for some $\beta \ge 0$,

$$\|Y_n t\|_p \ll 2^{\beta(\mu - n)} \|t\|_p, \quad \mu \ge n.$$

We give several examples of the sets of operators satisfying the conditions (A) and (B).

I. $Y_n = S_{2^n}$ is an operator that associates each function $f \in L_1$ with a partial sum of the Fourier series of degree 2^n. Then property (A) for $1 < p < \infty$ follows from the known results of approximation of functions from Sobolev classes by trigonometric polynomials of the corresponding degree (see, e.g., [5, p. 48]). By Theorem 1.1 in [5, p. 26], we can write

$$\|S_{2^n}\|_{p\to p} \le C_7(p), \qquad C_7(p) > 0, \quad 1 < p < \infty.$$

Hence, for an arbitrary $p \in (1, \infty)$, property (B) with $\beta = 0$ is true.

II. $Y_n = I_{2^n}$ is the operator of interpolation by trigonometric polynomials of degree 2^n at the nodes $\dfrac{2\pi l}{2^{n+1}+1}$, $l = 0,\dots,2^{n+1}$. It is known (see, e.g., [5, p. 86]) that, for these operators, relation (A) is true for $1 < p < \infty$ for $\rho > \dfrac{1}{p}$ and relation (B) is true for $1 < p < \infty$ with $\beta = \dfrac{1}{p}$.

In examples I and II, the case $1 < p < \infty$ is considered. We give one more example for cases $p \in \{1, \infty\}$.

III. $Y_n = V_{2^n}$ is a de la Vallée-Poussin operator of order 2^n. Property (A) for $1 \le p \le \infty$ follows from estimates for the best approximation of Sobolev classes (see, e.g., [5, p. 47]). Relation (B) is true for $1 \le p \le \infty$ for $\beta = 0$ (see, e.g., [5, p. 28]).

We define the operator $T_N,\ N \in \mathbb{N}$, acting on a function of d variables as follows:

$$T_N = \sum_{\mathrm{s}\in\kappa(N)} \prod_{i=1}^{d} (Y^i_{s_i} - Y^i_{s_i-1}), \tag{7}$$

where Y^i_n is the operator Y_n acting on a function of the variable x_i. Assume that $Y_{-1} \equiv 0$.

For the first time, operators of the form (7) were considered in [6]. For subsequent results concerning the investigation and use of operators of this type, see, e.g., [7–10]. In the case $Y_n = S_{2^n}$, the corresponding operators T_N were studied in [5, 11, 12] (see also the references in these works).

2. Approximation of Functions from the Classes H^Ω_p

We formulate and prove the following statement:

Theorem 1. *Let the operators $Y_n,\ n \in \mathbb{Z}_+$, satisfy conditions (A) and (B). Then, for any function $f \in H^\Omega_p$, $1 \le p \le \infty$, where the function $\Omega \in \Phi_{\alpha,l},\ \alpha > \beta$, and $l < \rho$, the error of its approximation by the operator T_N given by relation (7) is estimated as follows:*

$$\|f - T_N f\|_p \ll \frac{1}{N} (\log_2 N)^{d-1}.$$

Proof. For a fixed vector $\mathrm{s} = (s_1,\dots,s_d)$, we define the operator Δ_s acting from L_p into $L_p,\ 1 \le p \le \infty$, as follows:

$$\Delta_{\mathrm{s}} = \prod_{i=1}^{d} \Delta_{s_i}, \quad \Delta_n = Y_n - Y_{n-1}, \quad n \in \mathbb{N}, \quad \Delta_0 = Y_0.$$

Further, we define the Bernoulli kernel $F_\rho(\mathrm{x}),\ \mathrm{x} = (x_1,\dots,x_d)$, by the relation

$$F_\rho(\mathrm{x}) = \prod_{j=1}^{d} F_\rho(x_j)$$

and

$$\Delta_{\mathrm{s}} \mathbf{F}_\rho = \prod_{i=1}^{d} \Delta_{s_i} \mathbf{F}_\rho. \tag{8}$$

Then properties (A) and (B) lead to the following relations for the operators $\{\Delta_{\mathrm{s}}\}_{\mathrm{s}\geq 0}$ [9]:

(A′) $\|\Delta_s \mathbf{F}_\rho\|_{p\to p} \ll 2^{-\rho\|\mathrm{s}\|_1}$;

(B′) for an arbitrary trigonometric polynomial t of degree 2^{v_i} in the variable x_i, $i=\overline{1,d}$, for some $\beta\geq 0$ we have

$$\|\Delta_{\mathrm{s}} t\|_p \ll 2^{\beta(\|\mathrm{v}\|_1-\|\mathrm{s}\|_1)}\|t\|_p, \quad \mathrm{v}\geq \mathrm{s}.$$

Here and below, the inequalities $\mathrm{a}>\mathrm{b}$, where $\mathrm{a}=(a_1,\dots,a_d)$ and $\mathrm{b}=(b_1,\dots,b_d)$, mean that $a_i>b_i$, $i=\overline{1,d}$.

We show that, for each function $f\in H_p^\Omega$, $1\leq p\leq\infty$, the following representation is true:

$$f=\sum_{\mathrm{s}\in\mathbb{Z}_+^d}\Delta_{\mathrm{s}}(f), \tag{9}$$

where convergence is understood in the metric of the space L_p.

For $d=1$, property (A) implies that $\left\|f-\sum_{s=0}^{n}\Delta_s f\right\|_p\to 0$, $n\to\infty$, and, hence,

$$f=\sum_{s=0}^{\infty}\Delta_s f. \tag{10}$$

We now show that this decomposition holds for $d>1$. To this end, we estimate the quantity $\|\Delta_{\mathrm{s}} f\|_p$ from above. Since an arbitrary function $f\in L_p$, $1\leq p\leq\infty$, can be represented in the form [13, p. 304]

$$f=\sum_{\mathrm{v}\geq 1}A_{\mathrm{v}}(f), \tag{11}$$

where, for $f\in H_p^\Omega$,

$$\|A_{\mathrm{v}}(f)\|_p\ll\Omega(2^{-\mathrm{v}}),$$

according to the Minkowski inequality, we get

$$\|\Delta_{\mathrm{s}} f\|_p\leq\sum_{\mathrm{v}\geq 1}\|\Delta_{\mathrm{s}}A_{\mathrm{v}}(f)\|_p. \tag{12}$$

We estimate $\|\Delta_{\mathrm{s}}A_{\mathrm{v}}(f)\|_p, 1\leq p\leq\infty$. Let D_ρ denote an operator defined on a set of trigonometric polynomials that is inverse to the operator F_ρ. It is clear that this is a generalization of the operator of differentiation to the case of nonnatural ρ. Thus, we can write

$$\|\Delta_{\mathrm{s}}A_{\mathrm{v}}(f)\|_p=\|\Delta_{\mathrm{s}}\mathbf{F}_\rho D_\rho A_{\mathrm{v}}(f)\|_p\leq\|\Delta_{\mathrm{s}}\mathbf{F}_\rho\|_{p\to p}\|D_\rho A_{\mathrm{v}}(f)\|_p=\mathcal{J}_1.$$

By using property (A′) and the Bernstein inequality for trigonometric polynomials that, in terms of this notation, has the form

$$\|D_\rho A_{\mathrm{v}}(f)\|_p\leq 2^{\rho\|\mathrm{v}\|_1}\|A_{\mathrm{v}}(f)\|_p,$$

we extend the estimate for the quantity $\mathcal{J}_1$:

$$\mathcal{J}_1 \ll 2^{-\rho\|\mathrm{s}\|_1} 2^{\rho\|\mathrm{v}\|_1} \|A_{\mathrm{v}}(f)\|_p = 2^{-\rho(\|\mathrm{s}\|_1-\|\mathrm{v}\|_1)} \|A_{\mathrm{v}}(f)\|_p. \tag{13}$$

On the other hand, by the property (B$'$), for $\mathrm{v} \geq \mathrm{s}$ and some $\beta \geq 0$, we obtain

$$\|\Delta_{\mathrm{s}} A_{\mathrm{v}}(f)\|_p \ll 2^{\beta(\|\mathrm{v}\|_1-\|\mathrm{s}\|_1)} \|A_{\mathrm{v}}(f)\|_p. \tag{14}$$

Thus, according to (13) and (14), in view of relation (1), we find

$$\|\Delta_{\mathrm{s}} A_{\mathrm{v}}(f)\|_p \leq \min\left(2^{-\rho(\|\mathrm{s}\|_1-\|\mathrm{v}\|_1)}\Omega(2^{-\mathrm{v}}), 2^{\beta(\|\mathrm{v}\|_1-\|\mathrm{s}\|_1)}\Omega(2^{-\mathrm{v}})\right).$$

Returning to (13), we can write

$$\|\Delta_{\mathrm{s}}(f)\|_p \leq \sum_{\mathrm{v}\geq 1} \|\Delta_{\mathrm{s}} A_{\mathrm{v}}(f)\|_p \ll \sum_{\mathrm{v}<\mathrm{s}} 2^{-\rho(\|\mathrm{s}\|_1-\|\mathrm{v}\|_1)}\Omega(2^{-\mathrm{v}}) + \sum_{\mathrm{v}\geq\mathrm{s}} 2^{\beta(\|\mathrm{v}\|_1-\|\mathrm{s}\|_1)}\Omega(2^{-\mathrm{v}}) = \mathcal{J}_2.$$

Since the function Ω belongs to S^{α}, $\alpha > 0$, the function $\Omega(2^{-\mathrm{v}})/\prod_{j=1}^{d} 2^{-\alpha v_j}$ almost increases in each variable. Similarly, since the function Ω belongs to S_l, the function $\Omega(2^{-\mathrm{v}})/\prod_{j=1}^{d} 2^{-\gamma v_j}$, $0 < \gamma < l$, almost decreases in each variable. Hence,

$$\mathcal{J}_2 = \sum_{\mathrm{v}<\mathrm{s}} 2^{-\rho(\|\mathrm{s}\|_1-\|\mathrm{v}\|_1)} \frac{\Omega(2^{-\mathrm{v}})}{\prod_{j=1}^{d} 2^{-\gamma v_j}} \prod_{j=1}^{d} 2^{-\gamma v_j} + \sum_{\mathrm{v}\geq\mathrm{s}} 2^{\beta(\|\mathrm{v}\|_1-\|\mathrm{s}\|_1)} \frac{\Omega(2^{-\mathrm{v}})}{\prod_{j=1}^{d} 2^{-\alpha v_j}} \prod_{j=1}^{d} 2^{-\alpha v_j}$$

$$\ll \frac{\Omega(2^{-\mathrm{s}})}{\prod_{j=1}^{d} 2^{-\gamma s_j}} 2^{-\rho\|\mathrm{s}\|_1} \sum_{\mathrm{v}<\mathrm{s}} 2^{(\rho-\gamma)\|\mathrm{v}\|_1} + \frac{\Omega(2^{-\mathrm{s}})}{\prod_{j=1}^{d} 2^{-\alpha s_j}} 2^{-\beta\|\mathrm{s}\|_1} \sum_{\mathrm{v}\geq\mathrm{s}} 2^{(\beta-\alpha)\|\mathrm{v}\|_1}.$$

For $\beta < \alpha$ and $\rho > l$, we obtain

$$\mathcal{J}_2 \ll \frac{\Omega(2^{-\mathrm{s}})}{2^{-\gamma\|\mathrm{s}\|_1}} 2^{-\rho\|\mathrm{s}\|_1} 2^{(\rho-\gamma)\|\mathrm{s}\|_1} + \frac{\Omega(2^{-\mathrm{s}})}{2^{-\alpha\mathrm{s}}} 2^{-\beta\|\mathrm{s}\|_1} 2^{(\beta-\alpha)\|\mathrm{s}\|_1} \ll \Omega(2^{-\mathrm{s}}). \tag{15}$$

It follows from (12)–(15) that, for an arbitrary vector $\mathrm{s} \in \mathbb{Z}_+^d$, the order inequality

$$\|\Delta_{\mathrm{s}} f\|_p \ll \Omega(2^{-\mathrm{s}}), \quad 1 \leq p \leq \infty, \tag{16}$$

is true. In turn, it leads to representation (9).

Further, by using this representation, the notation $T_N f$, and the Minkowski inequality, we obtain

$$\|f - T_N f\|_p = \left\| \sum_{\mathrm{s}\in\mathbb{Z}_+^d} \Delta_{\mathrm{s}}(f) - \sum_{\mathrm{s}\in\kappa(N)} \Delta_{\mathrm{s}}(f) \right\|_p \leq \sum_{\mathrm{s}\in\kappa^{\perp}(N)} \|\Delta_{\mathrm{s}}(f)\|_p. \tag{17}$$

By substituting (16) into (17), by Lemma A and relations (5) and (6), we obtain

$$\|f - T_N f\|_p \ll \sum_{\mathrm{s}\in\kappa^{\perp}(N)} \Omega(2^{-\mathrm{s}}) \ll \sum_{\mathrm{s}\in\Theta(N)} \Omega(2^{-\mathrm{s}}) \asymp \frac{1}{N} \sum_{\mathrm{s}\in\Theta(N)} 1 \asymp \frac{1}{N}(\log_2 N)^{d-1}.$$

Thus, the theorem is proved.

Now let

$$\Omega(\mathrm{t}) = \omega\left(\prod_{j=1}^{d} t_j\right), \tag{18}$$

where ω is a given function of one variable of the type of modulus of smoothness of order l that belongs to the sets S^{α} and S_l. It is clear that the function Ω thus defined belongs to the set $\Phi_{\alpha,l}$.

Taking into account the special form of the function $\Omega,$ we rewrite operators (7) in the form

$$T_m = \sum_{\|\mathrm{s}\|_1\le m} \prod_{i=1}^{d} (Y^i_{s_i} - Y^i_{s_i-1}), \tag{19}$$

where $m \in \mathbb{N},$ according to (2)–(5), is determined from the relation

$$\omega(2^{-m}) \asymp \frac{1}{N}. \tag{20}$$

In [14], one more relationship between m and N

$$\log_2 N \asymp m \tag{21}$$

is established.

By using Theorem 1 and estimates (20) and (21), we arrive at the following statement:

Theorem 1′. *Let the conditions of Theorem 1 be satisfied and let the function Ω be defined by relation (18). Then, for any function $f \in H^{\Omega}_p$, $1 \le p \le \infty$, the error of its approximation by the operator T_m defined by relation (19) is estimated as follows:*

$$\|f - T_m f\|_p \ll \omega(2^{-m}) m^{d-1}.$$

3. Estimates for the Orthoprojective Widths of the Classes H^{Ω}_p in the Space L_p for $p \in \{1, \infty\}$

As a corollary of Theorem $1'$ and known results, we establish the order of orthoprojective widths for the classes H^{Ω}_p.

Recall that the orthoprojective width of a functional class $F \subset L_q$ in the space L_q is defined by the formula

$$d^{\perp}_m(F, L_q) = \inf_{\{u_i\}_{i=1}^m} \sup_{f\in F} \left\| f - \sum_{i=1}^{m} (f, u_i) u_i \right\|_q, \tag{22}$$

where infimum is taken over all orthonormal systems of functions $\{u_i\}_{i=1}^{\infty} \subset L_\infty,\ i = \overline{1, m}$. The width $d_m^{\perp}(F,\, L_q)$ was introduced by Temlyakov in [15].

Parallel with widths $d_m^{\perp}(F,\, L_q)$, we consider the quantities $d_m^B(F,\, L_q)$ also introduced by Temlyakov (see, e.g., [16]) and defined by the formula

$$d_m^B(F,\, L_q) = \inf_{G \in \mathcal{L}_m(B)_q} \sup_{f \in F \cap D(G)} \|f - Gf\|_q. \tag{23}$$

Here, $\mathcal{L}_m(B)_q$ denotes a set of linear operators G satisfying the conditions:

(a) the domain of definition $D(G)$ of these operators contains all trigonometric polynomials and their range of values is contained in a subspace of dimension m of the space L_q;

(b) there exists a number $B \geq 1$ such that, for all vectors $\mathrm{k} = (k_1, \dots, k_d)$, the inequality

$$\|Ge^{i(\mathrm{k},\cdot)}\|_2 \leq B$$

is true.

Since the operators of orthogonal projection onto subspaces of dimension m belong to $\mathcal{L}_m(1)_2$, according to the definition of the quantities $d_m^{\perp}(F,\, L_q)$ and $d_m^B(F,\, L_q)$, the following inequality is true:

$$d_m^B(F,\, L_q) \leq d_m^{\perp}(F,\, L_q). \tag{24}$$

The results of investigation of the quantities (22) and (23) for various functional classes can be found, e.g., in [12, 17, 18] and in the monographs [5, 16].

The following statement is true:

Theorem 2. *Let*

$$\Omega(\mathrm{t}) = \omega\left(\prod_{j=1}^{d} t_j\right),$$

where ω is a function of one variable that belongs to the set $\Phi_{\alpha,l},\ \alpha > 0$. Then, for $p \in \{1,\, \infty\}$, the following order estimate is true:

$$d_m^{\perp}(H_p^{\Omega},\, L_p) \asymp \omega(2^{-l}) l^{d-1}, \tag{25}$$

where $m \asymp 2^l l^{d-1}$.

Proof. First, we find the upper bound for (25). To this end, we take an arbitrary basis $\{P_k\}_{|k|=1}^{\infty}$ with the following properties:

(i) for any $|k| \geq 1,\ P_k(x)$ is a trigonometric polynomial of degree of at most $|k|$;

(ii) for any $k \neq l \in \mathbb{Z}\backslash\{0\},\ (P_k, P_l) = 0$ and $(P_k, P_k) = 1$;

(iii) $L_N = \max\limits_{x \in [0, 2\pi]} \displaystyle\int_0^{2\pi} \left| \sum\nolimits_{|k|=1}^{N} P_k(t) P_k(x) \right| dt \leq C_8,\ C_8 > 0$, for any $N \in \mathbb{N}$;

(iv) for any function $f \in L_p,\ 1 \le p \le \infty$,

$$\left\| f - \sum_{|k|=1}^{N} (f, P_k) P_k \right\|_p \le K E_{C_9 N}(f)_p \,,$$

where $K, C_9 > 0$ and $E_l(f)_p$ is the best approximation of the function f by trigonometric polynomials of degree of at most l in the metric of the space L_p.

For the examples of construction of these bases with the corresponding constants, see [19, 20].
We set

$$Y_n f = \sum_{|k|=1}^{2^n} (f, P_k) P_k, \quad n \ge 0, \tag{26}$$

and show that this sequence of operators $\{Y_n\}_{n=0}^{\infty}$ satisfies the conditions (A) and (B).

First, we show that the operators $Y_n,\ n \in \mathbb{Z}_+$, satisfy the condition (B) with $\beta = 0$. Consider the case $p = 1$ (for $p = \infty$, the proof is similar). Let t be an arbitrary trigonometric polynomial. Then

$$\begin{aligned}
\|Y_n t\|_1 &= (2\pi)^{-1} \int_0^{2\pi} \left| \sum_{|k|=1}^{2^n} (t, P_k) P_k(x) \right| dx \\
&= (2\pi)^{-1} \int_0^{2\pi} \left| \sum_{|k|=1}^{2^n} (2\pi)^{-1} \int_0^{2\pi} t(y) P_k(y) dy P_k(x) \right| dx \\
&= (2\pi)^{-1} \int_0^{2\pi} \left| (2\pi)^{-1} \int_0^{2\pi} t(y) \sum_{|k|=1}^{2^n} P_k(y) P_k(x) dy \right| dx \\
&\le \int_0^{2\pi} \left| \max_{y \in [0, 2\pi]} \left(\sum_{|k|=1}^{2^n} P_k(y) P_k(x) \right) (2\pi)^{-1} \int_0^{2\pi} |t(y)| dy \right| dx \\
&= \|t\|_1 \int_0^{2\pi} \left| \max_{y \in [0, 2\pi]} \sum_{|k|=1}^{2^n} P_k(y) P_k(x) \right| dx = \mathcal{J}_3.
\end{aligned}$$

By using property (iii), we complete the estimation of $\mathcal{J}_3$:

$$\mathcal{J}_3 \le \|t\|_1 \max_{y \in [0, 2\pi]} \int_0^{2\pi} \left| \sum_{|k|=1}^{2^n} P_k(y) P_k(x) \right| dx = L_{2^n} \|t\|_1 \ll \|t\|_1 \,.$$

Further, we show that the operators $Y_n,\ n \in \mathbb{Z}_+$, satisfy condition (A) with arbitrary $\rho > 0$. By using the estimates of the best approximation of functions from the Sobolev classes by trigonometric polynomials with the corresponding spectrum (see [5, p. 47]) and property (iv), we obtain

$$\|(I - Y_n)\mathbf{F}_\rho\|_{p\to p} = \sup_{\|\varphi\|_p\le 1} \|(I - Y_n)\mathbf{F}_\rho\varphi\|_p = \sup_{f\in W_p^\rho} \|f - Y_n f\|_p$$

$$= \sup_{f\in W_p^\rho} \|f - \sum_{|k|=1}^{2^n} (f, P_k)P_k\|_p \ll \sup_{f\in W_p^\rho} E_{2^n}(f)_p \ll 2^{-\rho n}.$$

We take $m \in \mathbb{N}$ and choose $l = l(m) \in \mathbb{N}$ such that $m \asymp 2^l l^{d-1}$. By Theorem $1'$, for any $f \in H_p^\Omega$, $p \in \{1, \infty\}$, the estimate

$$\|f - T_l f\|_p \ll \omega(2^{-l}) l^{d-1}$$

is true. It follows from (26) that $T_l f$ is an operator of taking partial sums of the Fourier series in the system $\{P_{\mathrm{k}}\}_{|\mathrm{k}|\ge 1}$, where

$$P_{\mathrm{k}}(\mathrm{x}) = P_{k_1}(x_1)\dots P_{k_d}(x_d).$$

According to the definition of orthoprojective width, we have

$$d_m^\perp(H_p^\Omega, L_p) \ll \sup_{f\in H_p^\Omega} \|f - T_l f\|_p \ll \omega(2^{-l}) l^{d-1}, \quad p \in \{1, \infty\},$$

where $m \asymp 2^l l^{d-1}$.

The lower bound in (25) follows from inequality (24) and the results obtained in [21].

The theorem is proved.

Remark 1. For $\Omega(\mathrm{t}) = \prod_{j=1}^d t_j^{r_j}$, $r_j > 0$, $j = \overline{1, d}$, the statements similar to Theorems 1 and 2 were established in [9].

Remark 2. Theorem 2 complements the results obtained in [21, 22].

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